Linear Dynamical Systems
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Week - 01
State - space solution and realization
Lecture - 05
Realization of LTI and LTV Systems

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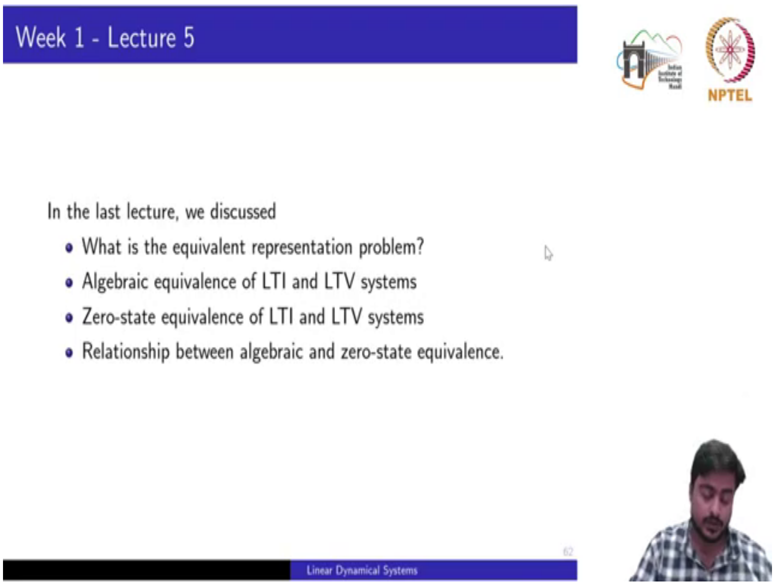

So, hello everyone. So, today we will see the lecture 5 of the week 1. So, in the last lecture we discussed that what is the equivalent representation problem, we discussed the notion of the equivalent representation. We also discussed the notion of algebraic equivalence, the zero-state equivalence and also the relationship between the zero-state equivalence and algebra equivalent. One of the most important result what we had discussed that time varying state matrix was transformed into a constant matrix right.

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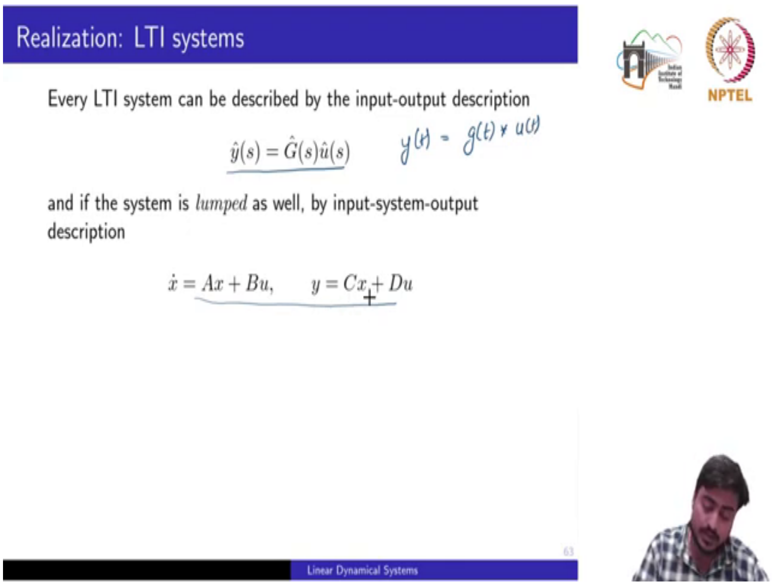

So, today we will discuss the realization problem of the, first we will, again we will discuss the about the LTI systems and then we will move towards to the LTV case. So, first we will see that how do you differ, what is the problem statement for the realization. So, we know that every LTI system can be described by the input output description given by this one, which is also the zero state response.

And if the system is lumped, by a lumped I mean to say that the system is having a finite set of state vectors variables as well by in input state output description which is given by this one. Meaning to say that this representation or when we represented as y of t as the convolution on g t in to u t. It can also be used to describe a distributed system right, but if we are having alarm system then it can be represented also by the state space equation ok.

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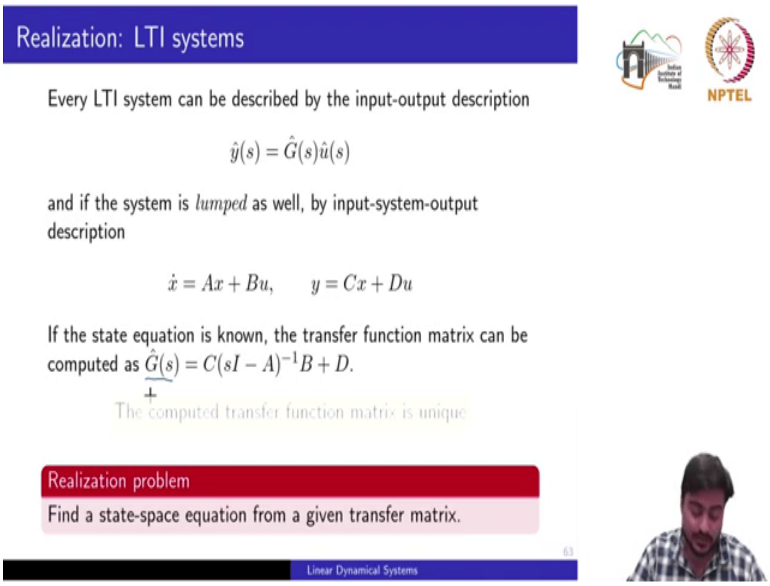

So, we also know that if the state equation is known, the transforming function matrix can be computed as this one, and this we also had seen in the last lecture about their equivalence, in the algebraic equivalence, in the zero state equivalence ok. Now this realization problem deals with the converse problem.

So, we know of n, in fact, we had seen that the computed transfer function matrix is unique in the sense. So, try to recall that in the last lecture we discussed to, at least two different representation of the same system right, but the transfer function was same for both the representation, meaning to say that there the transfer function is unique, but not the state space representation.

So, this realization problem deals with the statement that find a state space equation from a given transfer function ok. Then let us say some G hat of s is given to us, now we know that

G hat is a unique description of the system. Now for the unique description we need to identify one state space representation and there could be infinite number of state space representation. So, we need to address the problem of computing the state space equation from a given transfer matrix.

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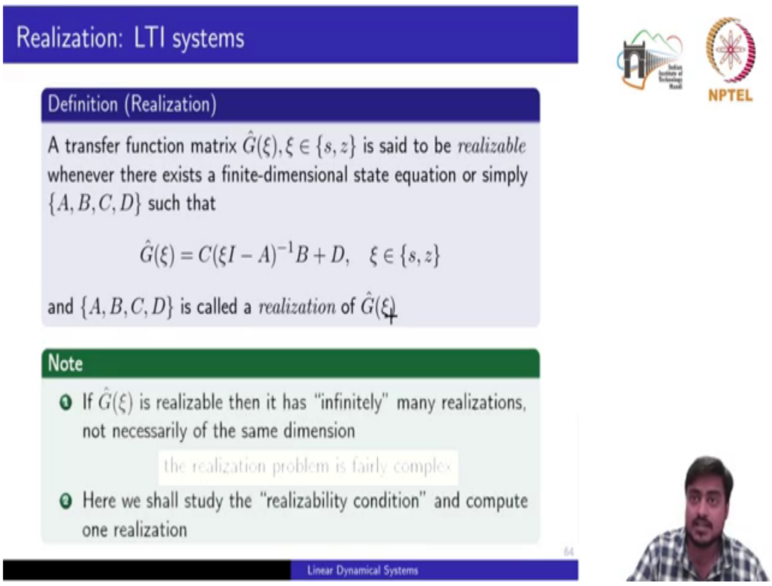

So, let us define it formally that a transfer function matrix G hat of xi. So, I use this variables xi just to denote that it is this definition is valid for the continuous time system and for the discrete time system. So, in the continuous time we go with the Laplace transform or the Laplace operator s and in the discrete time system we go with the z transform operator by z, that the transfer function matrix G hat is said to be realizable whenever there exists a finite dimensional state equation or simply the pair ABCD such that this equation is satisfied, and we call this ABCD is the realization of G hat of s or xi ok.

So, we have already discussed that G hat of xi could have infinite number of realization in the sense that there could be infinite number of the pair ABCD. So, this problem is fairly complex that among all those representation which representation is to choose or how to ensure that there exists a representation right. So, in this topic we will study the realizability condition and we compute only one realization. So, given a transfer function matrix we will study those conditions under which we can say that this transfer function is realizable and with respect to that we compute one adhoc realization ok.

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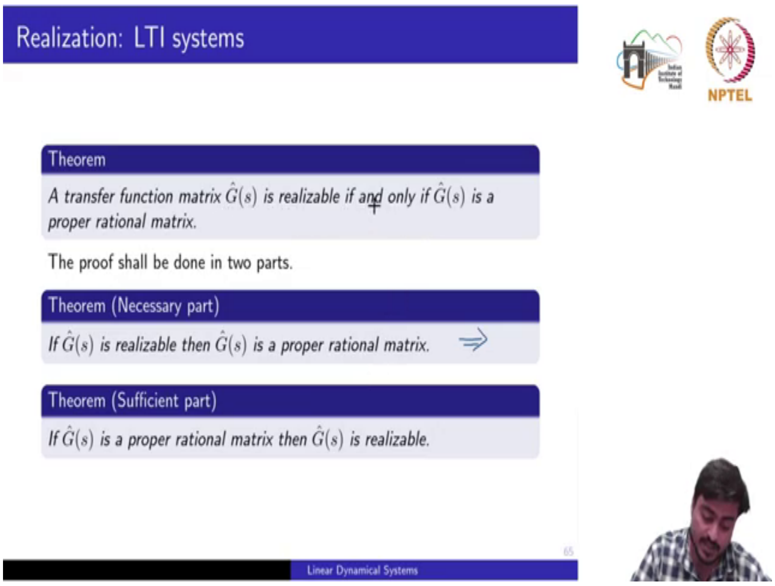

So, this is the most important result that a transfer function matrix G hat is realizable, if and only if G hat of s is a proper rational matrix. So, in fact, this point we also mentioned in the beginning of this week 1 when we were discussing about the causality and all. So, the proof of this result is quite long and we will give the proof in two different part which first we call the necessary part and the sufficient part.

So, in the necessary part that if G hat is realizable then G hat is a proper rational matrix, basically we also denoted by this symbol. Meaning to say that the realizability of G hat of s implies that G hat is a proper rational matrix ok. The sufficient part which is the converse of this part that if G hat of s is a proper rational matrix then your G hat of s is realizable. If this necessary insufficient condition is satisfied then the realizability of G hat of s is basically equivalent to saying that G hat is a proper rational matrix.

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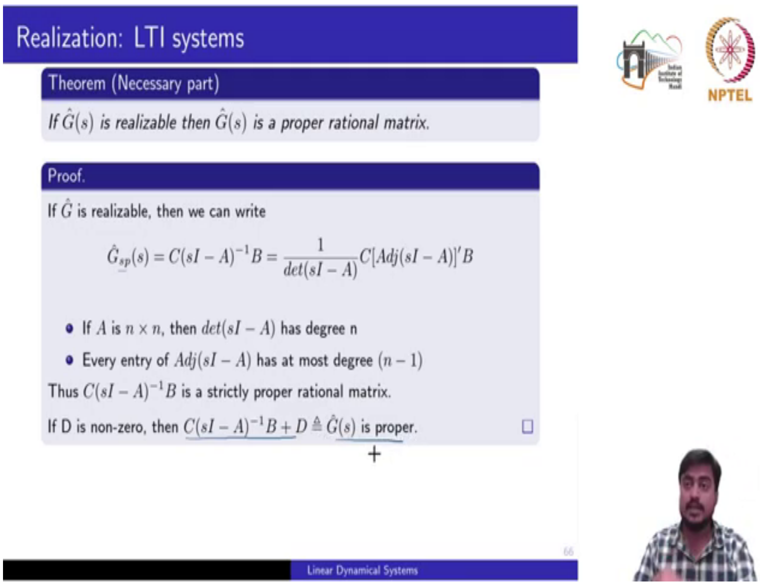

So, let us see the first the necessary part. So, we already assumed that G hat of s is realizable. So, if g hat is realizable then we can write it by this part ok. Let us denote this G hat underscore sp, where I have not included the d matrix. So, without the d matrix we know that see that they defined the transfer function ok, and this transfer function we call it the sp. And we know that I can define this sI my, the inverse of sI minus A by the ratio of adjoint sI minus A transpose divided by the determinant of sI minus A. Since it is a polynomial I can commute ok.

Now, note here that if A is a n cross n matrix then the determinant of sI minus A would have the degree of the characteristic polynomial equal to n right. We also discuss in the in one of the previous lecture that the degree of this adjoint of sI minus A would be a at most s minus 1, sorry n minus 1.

So, we know that this part see sI minus A inverse B is a strictly proper part this is the reason we defined it sp which implies the strictly proper part. Now if the d matrix is a nonzero matrix then this whole part is basically the proper. So, this shows the proof of the necessary part that if G hat is realizable, which we started from here then G hat of s is a proper rational matrix, and we arrived to the conclusion that G hat is a proper rational matrix. The sufficient part which is the converse.

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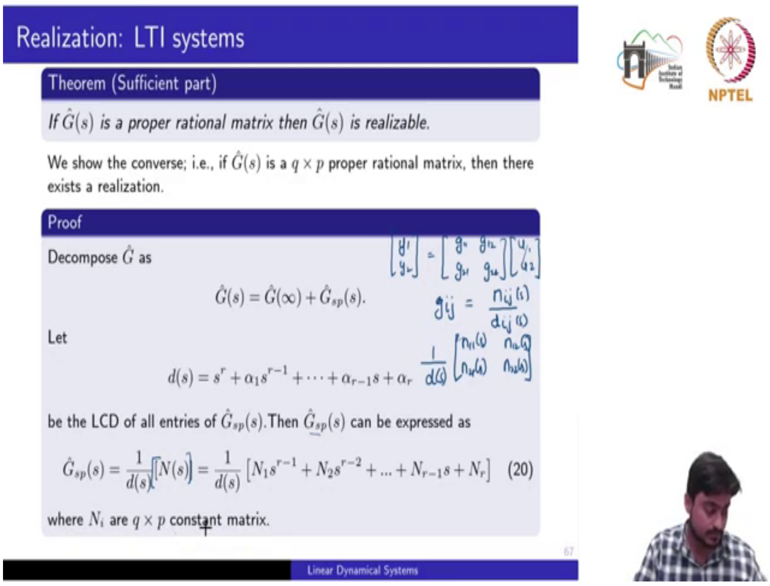

So, we show the converse that is if G hat is a q cross p proper rational matrix, then there exists a realization, meaning to say if there exists a realization then that transfer function is realizable. So, let us see so, first of all we decompose the transfer function G hat has the summation of G hat infinity plus the strictly proper part of the G hat and we know that this G hat is basically nothing, but the d matrix ok.

Let define this polynomial this d of s with whose order or degree is r, which is basically the least common denominator of all the entries of this strictly proper part of the transfer function ok. Then I can express this Gsp as the ratio of 1 by d s, because d of s is a polynomial and this is basically a matrix which is n in n the function N of s, and this N of s is basically given by this long part where we have r number of constant matrices of dimension q cross p ok.

In all these s are the scalars. So, let us say if you want to visualize, let us say I have a 2 by 2 system ok. So, this 2 by 2 system I can represent by y 1 y 2 equal, let us say g 1 1, g 1 2, g 2 1 and g 2 2. Let us say we have this 2 by 2 system and these or this g 1 1, g 1 2, g 2 1, in g 2 2 add that transfer function basically the ratio of polynomials.

Now, this I can express this, let us denote this gi gi j, let us say some polynomial j and some denominator polynomial i j of s. So, this after taking the LCD of all the elements of this transfer function matrix I can express this transfer function matrix is let us say some LCD of this matrix 1 by d s then this would be a matrix in the sense n 1 1 of s n 1 2 of s like this.

So, the same has been expressed in this one ok, because all this n ij would be the function of s and I can take the N matrix common and finally, all this N I s would be the constant matrix of dimension q cross p ok.

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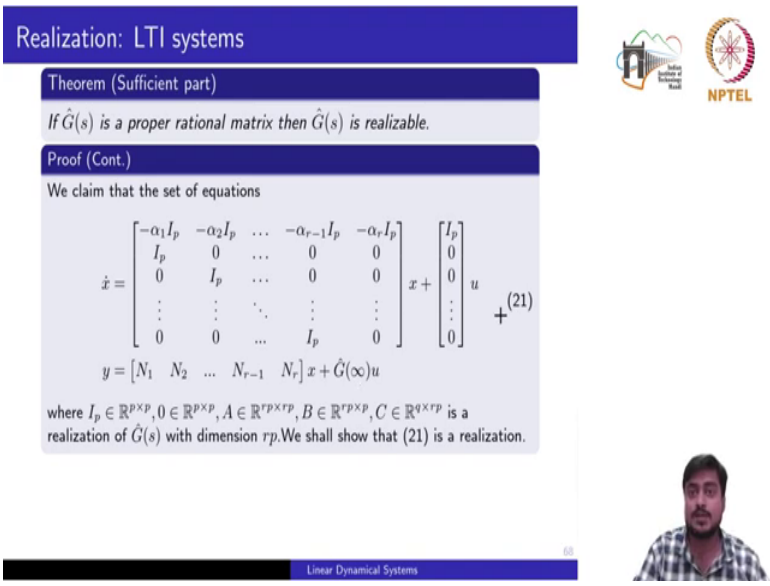

So, now we give a set of equation given by this A matrix this B matrix this C matrix in this D matrix and we claim that this set of equations is a realization of the transfer function G hat of s, where all these Ip are the identity matrix of dimension B. All these 0s are also the matrices having its elements all its elements 0, this A matrix is having a dimension of rp into rp ok.

So, note that here its that what the point what we had raised earlier that there could be infinitely number of state space representation, and it is not necessary that all these representations would be having the same dimension right.

So, here we would be having this the A matrix or dimension r p into r p, b r p into r p and c q into r p in overall it would give me the transfer function of dimension q cross p. You can verify by yourself the these dimension of these matrices, because we have r number of elements of p matrices which are the block matrices. So, in overall it will give me this dimension ok.

So, now we need to show that this set of, this state space representation is a realization; is a realization. If we are able to show this, meaning to say that G hat of s is realizable transfer function.

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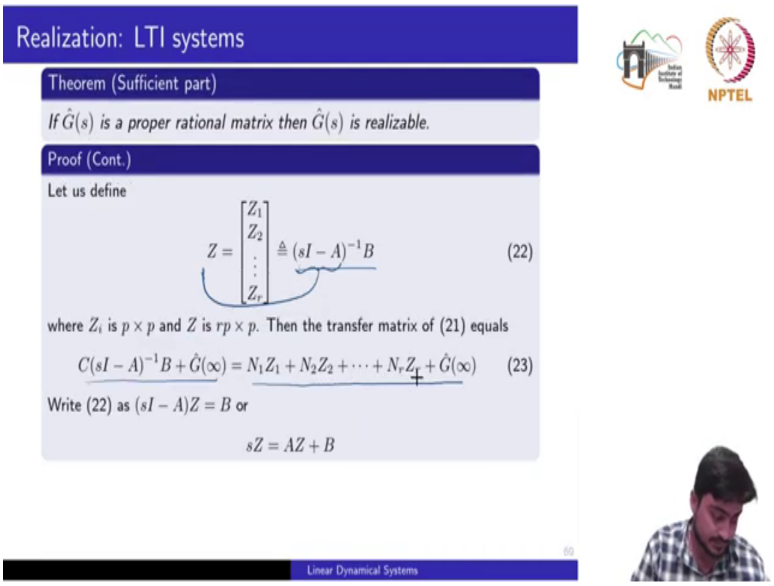

So, let us see, so first of all we define this Z as Z 1 2 Z r equal to s the inverse of sI minus A into p where all these Zi are the p cross p matrix. So, if all these Z is are the p cross p matrix, the dimension of this matrix capital Z would be rp into p because we have r number of elements of matrices p cross p.

So, if I write that transfer function of the state space representation what we had introduced earlier, which in general way I can represent by this is given by this, where I have replaced all this sI minus A inverse B by there Z 1 to Z r ok, you can write it more explicitly to finally, arrive to this equation right. This equation I can also write it sI minus A into Z is equal to B by taking this part or pre multiplying both side by sI minus A ok, sI minus A Z into B or s Z is equal to AZ plus B.

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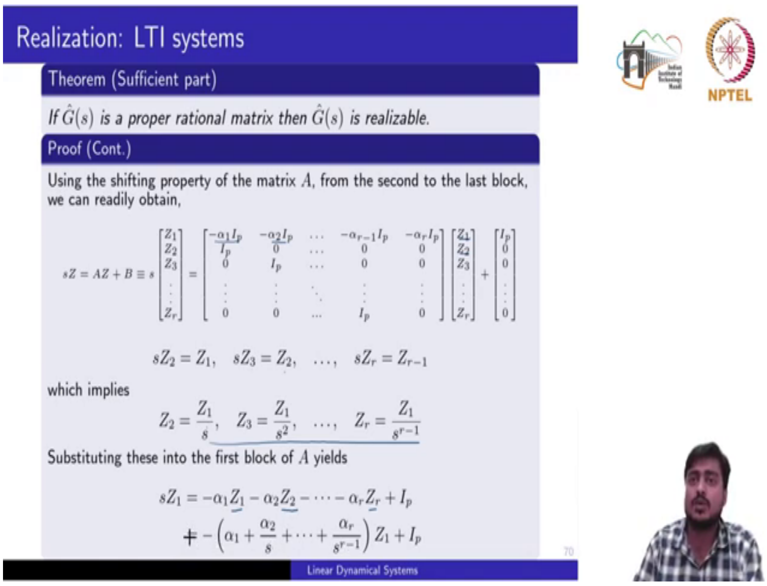

So, let us see this part s Z is equal to A Z plus B, where now I have explicitly written this Z matrix starting from Z 1 to Z r. We have this A matrix into Z plus B matrix of the state space equation what we had introduced. So, see from starting from the second element to the last element, so if you notice we have Z 2 is equal to or s into Z 2 is equivalent to Ip into Z 1.

So, this is this one that s times Z 2 is equal to Z 1 if we see the third element which is s times Z 3 is equal to I p times Z 2 where I is an identity matrix and so forth and so on, if we keep on continuing it could be s into Zr is equal to Ip times Zr minus 1 ok, because all these elements are 0.

Now, from here we compute this, we can write Z 2 s Z 1 by s Z 3 is Z 2 Z 2 by s and then replacing Z 2 by Z 1 by s we would get, we can parameterize all these Z in terms of Z and the degrees of s. And after replacing all this Z putting all these Zs into the first equation which is s times Z 1 is equal to this into this plus this into this plus Ip.

This is the part in replacing all this Z 2 2 Zr by these equations; in fact, we have parameterized this whole equation in terms of Z 1 only. You can write this equation by yourself as well to visualize that how we have arrived to this part.

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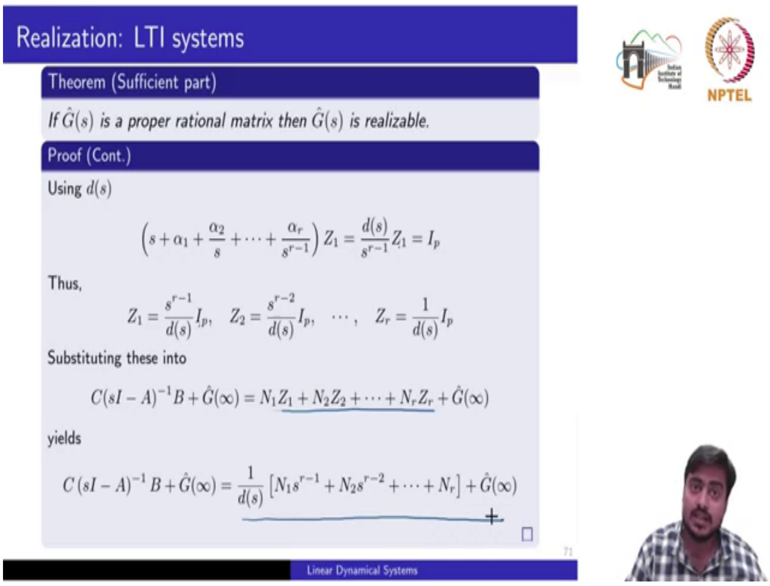

Now, using the definition of d of s what we have introduced, I can write this last part. If you pay attention to this one this part, so this part I am trying to simplify it in the next line by this one. So, I can write it is the ratio of the polynomial d of s divided by the s of power r minus 1 into Z 1 is equal to Ip ok. So, once I have defined Z 1 which is s to the power r minus 1 divided by d of s into Ip, I can compute all these the remaining Z s which were parameterize in terms of Z 1 ok. Now substituting all these Z s into the original equation, into the original equation if you see here this part from where we started.

So, after putting here I get the original transfer function of the state space system. Meaning to say that the state, the set of equations what we have introduced is one of the realization of the transfer function ok, this proves the sufficient part. So, we are now sure that the realization of the G of hat G hat of s is basically equivalent to saying the G hat of s should be a proper rational matrix ok.

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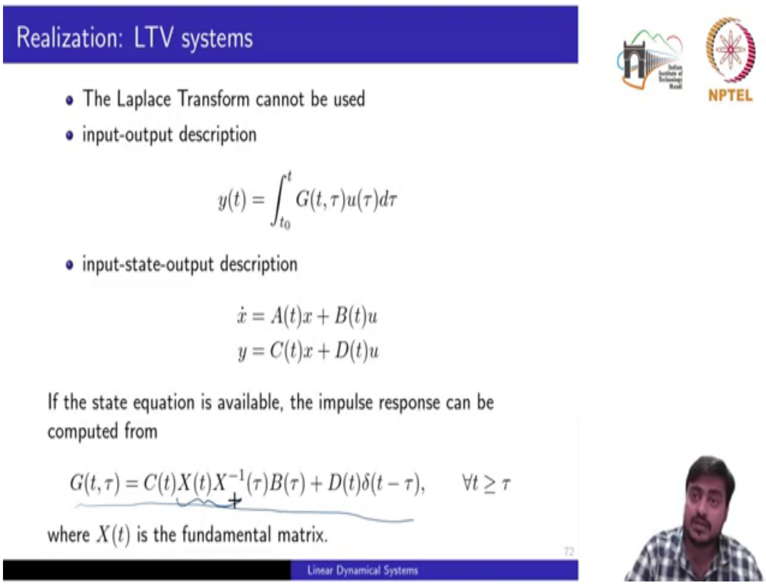

Moving on to the LTV systems. So, we know that we cannot use the Laplace transform tool, for the two basic reasons what we have introduced and the one of the previous lectures. So, the input-output description is given by this and the input state output state description is given by this. Again recalling the same problem that if the state equation is available the impulse response can be computed from this one, where this part is basically the state transition matrix, the solution of the state transition matrix in terms of the fundamental matrix ok.

Again we want to answer the reverse problem that given a transfer function, I want to compute the pair ABCD; the all that time varying parameters.

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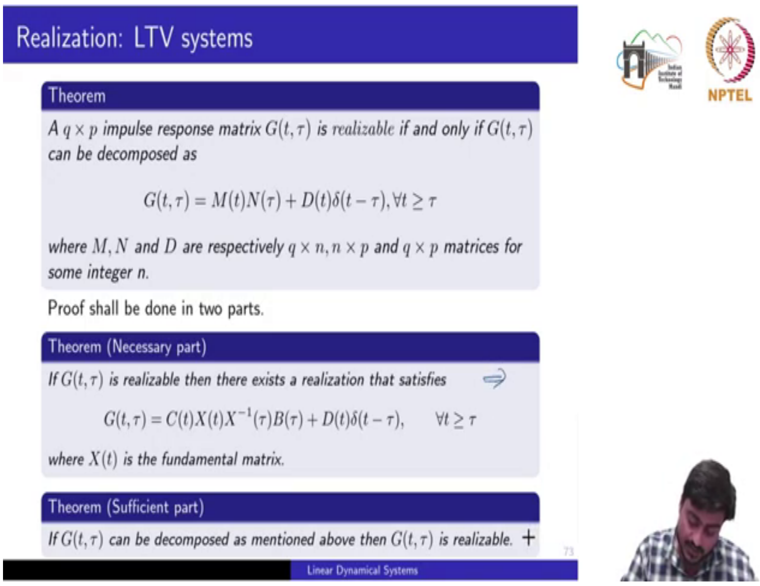

So, this is the result for the LTV system that given A q cross p impulse response matrix G t comma tau, we say that it is realizable if an only if that matrix can be decomposed by this where M N D are respectively the matrices of appropriate dimensions. Now, that this result says that if any transfer function matrix is for the transfer or the impulse response matrix. I cannot use the word transfer function, because for the LTV systems we do not have the transfer functions.

So, for given impulse response matrix Gt comma tau, we say there exists a ABCD matrices, a the given response matrix can be decomposed into this part. Again we will be doing this the proof of this result into two parts; one is the necessary part another is a sufficiency part that if Gt comma tau is realizable, then there exist a realization that satisfy this one, this was the idea of reducing here.

Because if it is realizable or the state equation is real available then I can express the impulse response matrix by using the parameters of that state equation, meaning to say that there exists some parameters. So, there exist a realization that satisfies this equation, where X is the fundamental metrics and the sufficiency is speaks about that if G t comma tau can be decomposed as this then the response matrix is realizable, both ways; this way and similarly this way ok.

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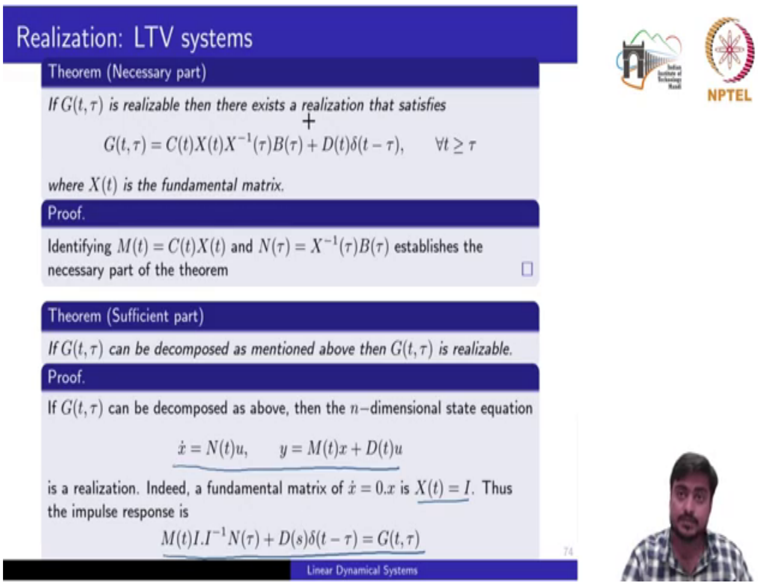

So, let us see the proof of the necessary part. So, this proof is pretty much straightforward that pay attention to here, this transfer the sorry the response matrix. So, if we denote this CX by M in this part is N right. So, we have in fact, already decomposed the transfer function in terms of the required result, meaning to say that Gt comma tau is realizable then there exists a real realization they satisfy this part.

So, M t and N tau we can replace this and this satisfy the necessary part of the theory ok. The sufficiency part that if G; the impulse response matrix is decomposed as mentioned above then it is realizable, we have the proof of this one that if Gt comma tau can be decomposed as above then the n dimensional state equation is given by this ok.

So, here we are introducing a state space representation of the time varying matrix. Note that the state matrix is 0, there is no matrix which is being multiplied by the state variable. So, we have x dot is equal to N of t into u and the output equation is given by M of t into x plus D of t into u right. So, if we compute the fundamental matrix of this one, we already know that the fundamental matrix give is a identity matrix right.

And thus the impulse response, if I compute the impulse response of this one is given by finally, the impulse response matrix, which proves the sufficiency part of this theorem, meaning to say that decomposing. So, if we go to the result that decomposing the given impulse response matrix into this form, is equivalent to saying that the response matrix is realizable, meaning to say that there exists the pair ABCD ok. So, here we conclude the theoretical part of the first week. So, in the next lecture we will see the, to do some tutorial problems.