Linear Systems Theory
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Module - 04
Lecture - 02
Introduction to Linear Systems

Welcome to this 2nd lecture of week 4 where we will talk about solutions to forced differential equations and how to generalize that to a state space system or a vector differential equation or general state space of n-dimensional systems model by $\dot{x}$ =Ax+Bu.

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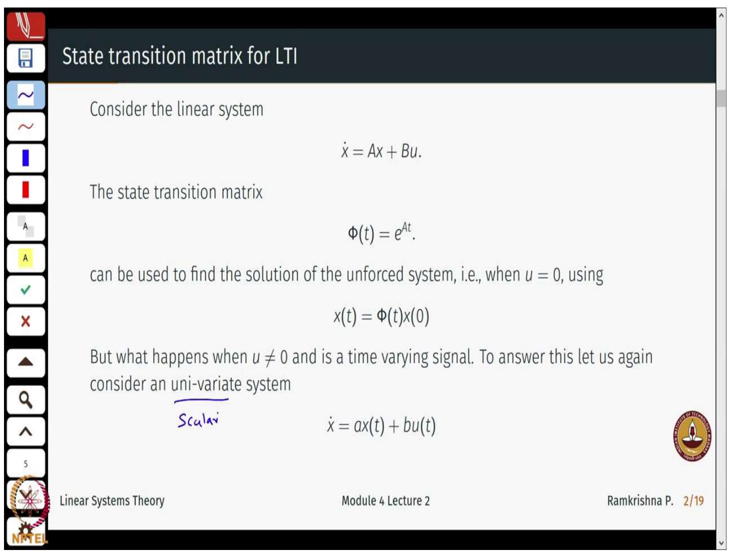

Let us start with a Linear System $\dot{x}$ =Ax for the moment, well let us left leave out B and what we had last time was that the state transition matrix, starting at time t equal to at the, initial time t ≠ 0 was given by e^{At} right. And then so we said that the solution x(t) is simply Φ(t) x_0.

So, this comes gets us to the next set of question that we should answer that again would be very instrumental in deriving future results in this course on controllability and observability and related concepts. So, what happens when u ≠ 0 and u is. Well it may not necessarily be a constant, but can also be time varying ok.

So, as usual just to make things a little easier we start with a univariate or even so called as a scalar system.

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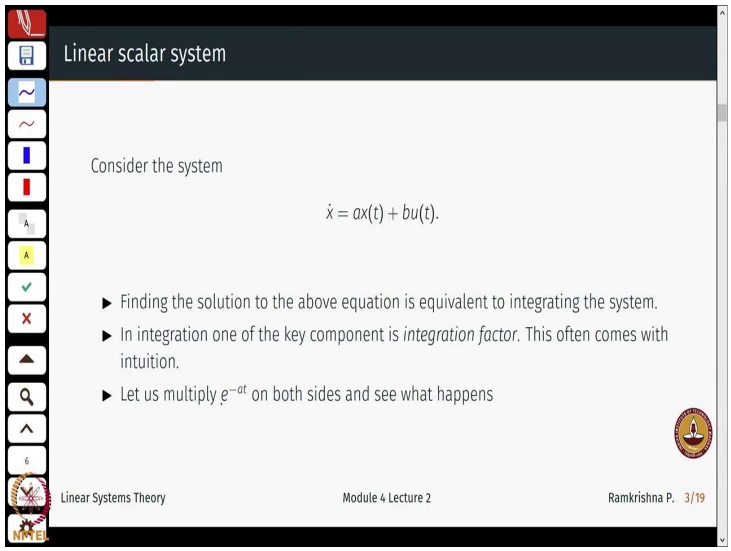

So, I just say $\dot{x}$ =ax+bu and finding solutions is as usual again integrating the system ok. So, just keeping in mind or just following some of the tricks that we used to solve equations of this kind, just say I multiply both sides by e^{at} and do all the relevant steps now.

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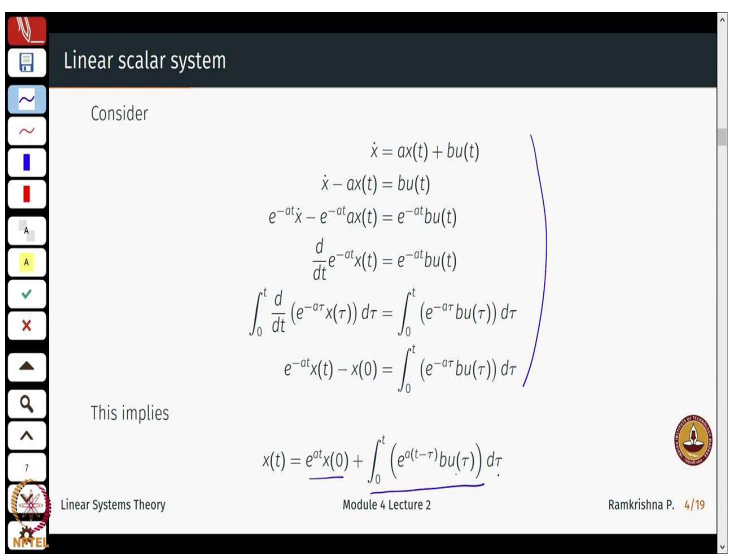

So, I will just keep all this, keep reading all the steps, it is very self explanatory. So, by doing all this it implies that x(t) is $e^{at}\ x_o$. This is again the solution to the unforced system plus an extra term now which comes because of the input at t - τ, sorry $e^{a(t-\tau)}$bu(τ)dτ ok. So, this is again something which we can kind of compute by using some kind of intuition that we know from our earlier courses in calculus.

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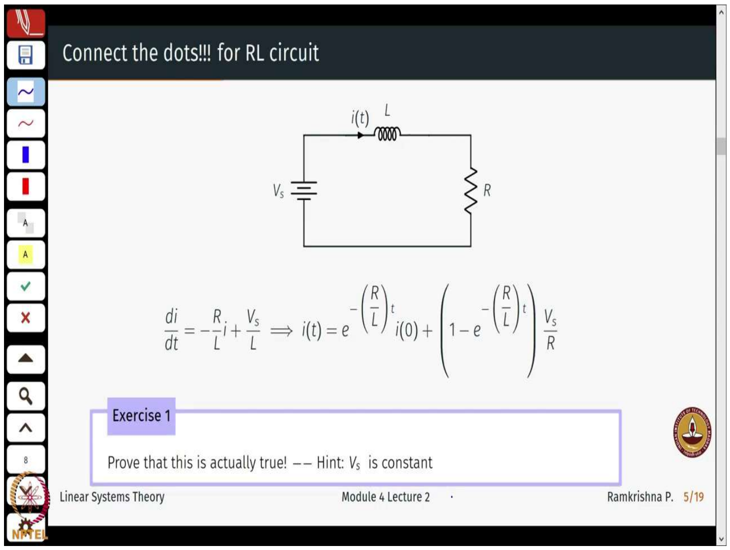

And that is what happened in the example that we also checked earlier, right. You can use this method or even directly of how we computed by using Laplace transforms for example, ok.

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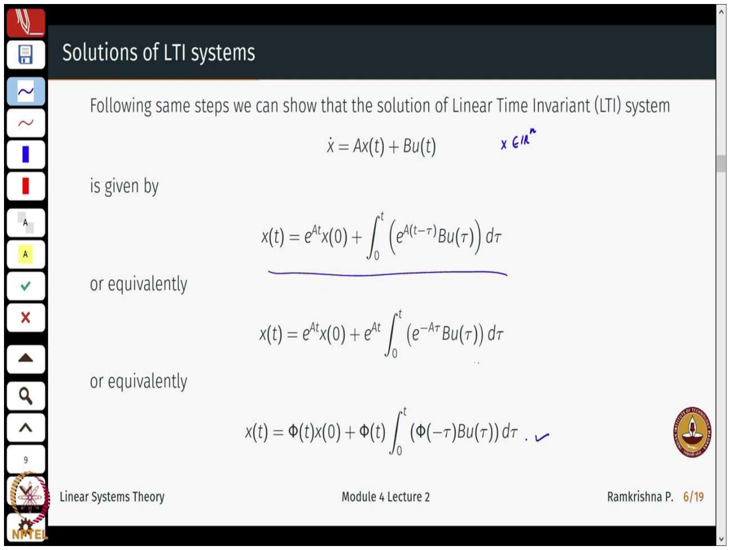

So, following this similar steps, now can we show that the solution of an LTI system, where x is now n-dimensional state vector is given by this particular expression or equivalently you know since I can just get this e^{At} outside and that the equation looks something like this. Or in general if I were to write it has a state transition matrix Φ (t), then the expression would look something like this ok. So, let us see if this is actually true.

So, where do I start from? I start from ok; there is some rule which I will use in the proof. I start from $\dot{x}$ is Ax + B u.

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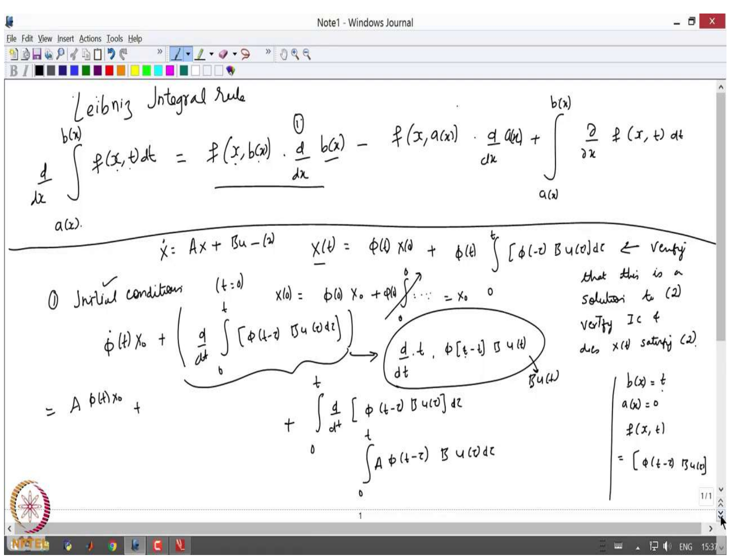

Again we just refer for simplicity or just, this is the beginning of this lecture, so we will just again look at time invariant case and we will explicitly deal with the time varying case in the next lecture ok

So, what do I propose as a solution to this set of equations is the following; x(t) and let it in a bit of a general form is Φ(t) x_0 + Φ (t) $\int_0^t \Phi(-\tau)$B u(τ)dτ, ok. Now the question is let me call this equation number 2 and just to say verify that this is a solution to the differential equation labelled as 2 ok.

So, how do we satisfy, how do I verify this first? Check initial conditions, ok. So, what happens at t = 0? t = 0, so on the left hand side I have x(0), here I have Φ (0)x_0 which is the initial condition plus integral; so, then Φ(0) integral from 0 to 0 and probably in the integral. So, the reason I do not write anything here is, because I am just integrating from 0 to 0, so this term will vanish and I am just left with X naught ok. So, the initial conditions are fine ok.

Now the second thing is thus. So, first is verify the initial conditions and second now thus this X of t satisfies the differential equation 2, ok. So, I just differentiate this and put it back. So, I have; so the first term $\dot{\Phi}$(t) x_0 + $\frac{d}{dt}$: 0 to t ok, I just get this inside for a while Φ(t-τ) Bu(τ)dτ ok.

So, the first term $\dot{x}$ is Φ(t) I know is AΦ(t) with x_0 plus ok. Now I have a strange looking expression here that which I want to differentiate with the integral sign. So, I just use the Leibnitz integral rule right, which we possibly would remember from some of our earlier courses ok. So, I have a function here, you know f (x,t) integrating from a(x) till b(x) and then the formula is given by this one, ok.

So, let us start with this side so, just I am used to concentrating on the right hand side, on this term which is differentiation with the integral sign here. So, the first term will be, you have a $\frac{d}{dt}$. So, this is my b(x). So, let us just write down for the simplicity what is. So, b(x) is my t, a(x) is 0, f(x,t) is Φ(t − τ) Bu(τ). So, this will be Φ(t -t)B u(t) ok.

So, this is what the first expression we will look like. Second expression a is 0, so this will not come into our case and the third term would be the integral of again some $\frac{d}{dt}\int_0^t$ AΦ(t − τ) Bu(τ)d τ. So, this is equal to, so just this term inside the inside the integral. So, what does what is the first term here. So, $\frac{dt}{dt}$ is 1 Φ(t -t) is the identity so, I am just left with b u (t) right. So, this term here.

This is the simplification of the first term here. Second term is 0 and I am looking now at the third term. So, the third term would simplify to $\frac{d}{dt}(\Phi(t))$ this will be $\int_0^t A\Phi(t-\tau)\, Bu(\tau) d\,\tau$ ok. So, let me again write this again. So, what we have so far. So, $\dot{x}$ so, sorry I am just differentiating this term, so what do I have here?

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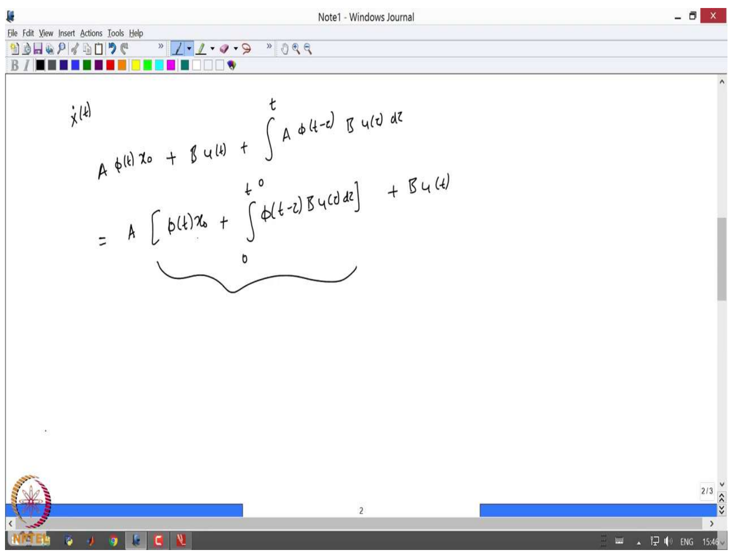

So, I have $A\Phi(t)x_0 + Bu(t) +$ the remaining term here. So, I have $\int_0^t A\Phi(t-\tau)\, Bu(\tau) d\,\tau$ ok. So, I can rearrange this terms as $A\Phi(t)x_0 + \int_0^t \Phi(t-\tau)B\, u(\tau)d\tau + Bu(t)$. Now, what is inside here? $\Phi(t)x_0$ + this term in the integral, this is exactly the solution that I am proposing here right, this one ok.

And therefore, I just I verify two conditions here that my solution satisfies the initial condition, it also satisfies the differential equation labelled as two. And therefore, this expression here is a solution to $\dot{x}$ = Ax + Bu. Again i use the notation Φ because I will use. So, this will be easier when I when use the same notation for linear time varying systems. Whereas, just as a little exercise you can do the same proof verifying the initial conditions and the differential equation for a specific type of a state transition matrix; that is e^{At} which is valid for LTI systems ok.

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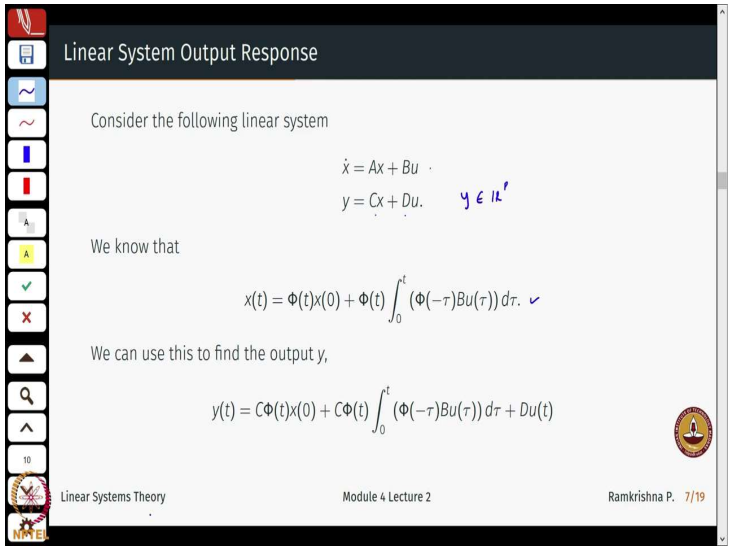

So, once I know this I can compute what is x (t), not only that if I have the expression for output where y could be some p-dimensional vector and then C and D being of appropriate dimensions. Y can be computed once I know the value or the solution to x ok.

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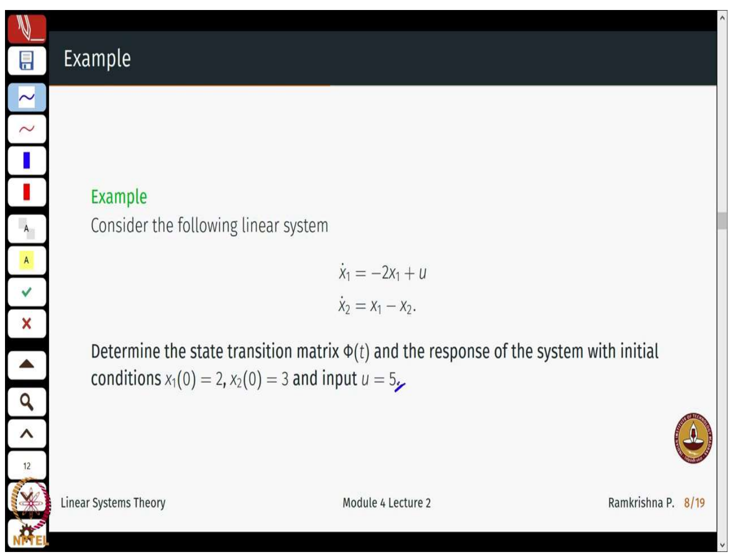

So, I can I just take a simple example and I just maybe compute for u = 5.

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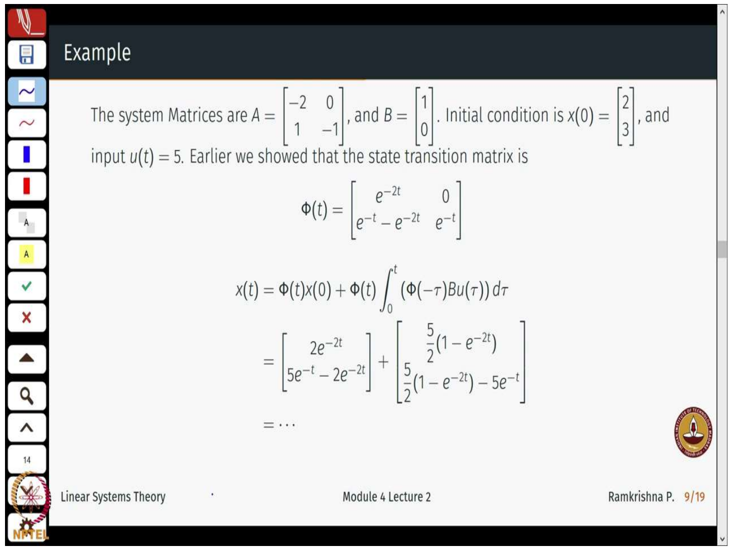

I just keep the step straight and they are, written down very neatly here and you can just write it down for yourself and verify it, You just an exercise some mechanical laborious exercise. So, I will just keep reading this example. If you have any questions, you could always post them on the forum ok.

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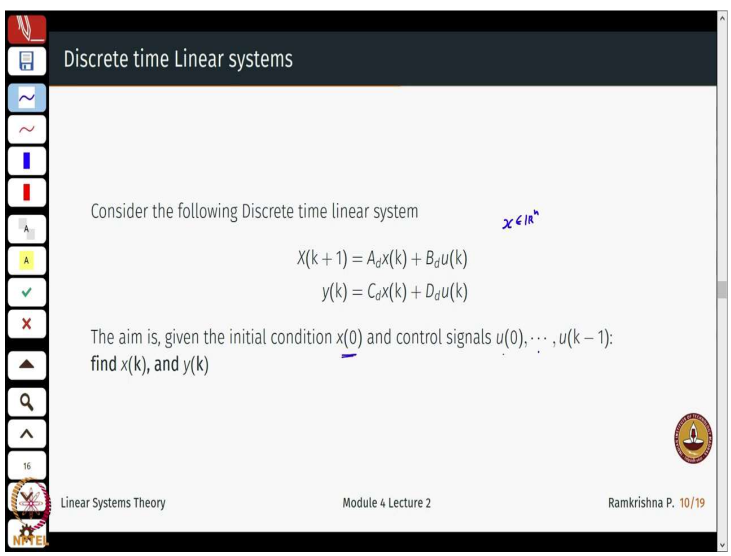

So, one; so, before we conclude this lecture we just do a little introduction to discrete time system. So, as said in the equation, so again x will be your n dimensional vector and so on; ABC and D will be the same kind of matrices that were in the continuous time system.

So, what is the aim here, right? So, if I what does it mean by even finding solutions, there will be. So, a given initial condition x_0 and control signals u(0) till u(k-1), can I find x(k) and y(k), right the state or the solution to this differential equation ok.

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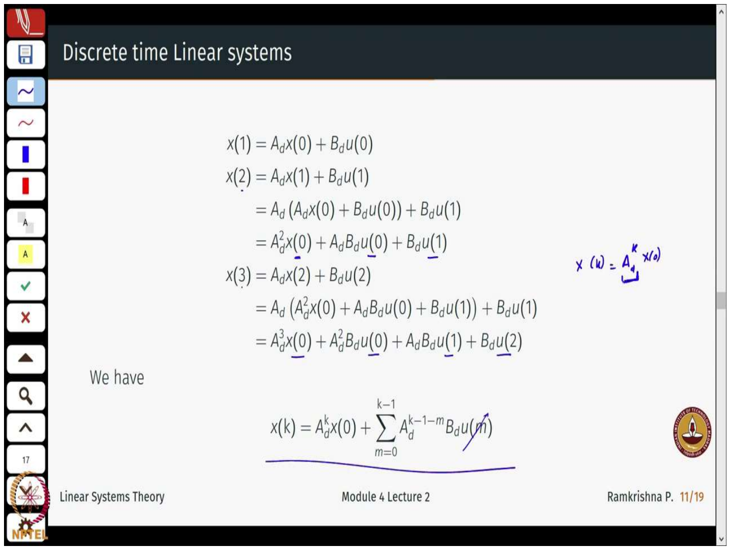

It turns out to be simpler, I will just write down these steps and some much of this should be like very obvious to us.

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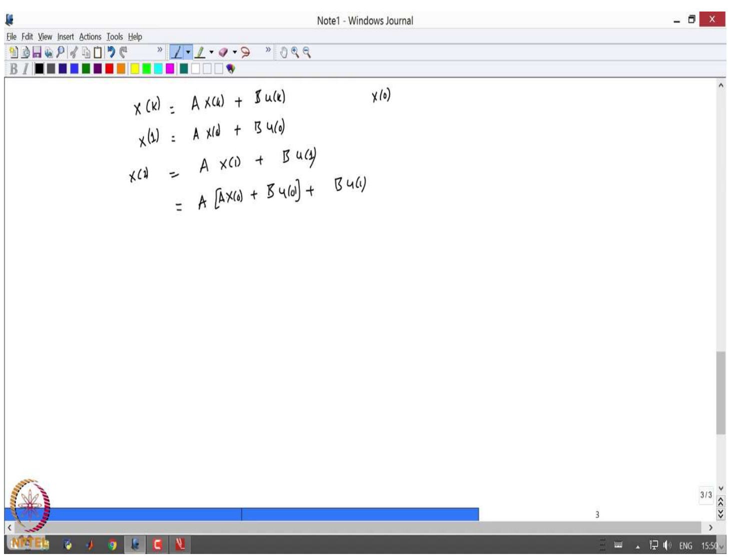

So, I have x(k) is Ax(k); again I am assuming that A and B are constant, even though we will in the next lecture maybe look at time varying case ok so, given is certain initial condition x_0 ok. Given x_0 ok, so what how can I find x(1)? x(1) is Ax_0 + Bu_0 ok. Similarly x(2) = Ax(1) + Bu_1, what is x(1)? x(1) is x_0 +Bu_0 + Bu_1 ok sorry it will be a here it. And therefore, so I will have a square term depending on x_0. So, to compute x_2 I need x_0, I need the input at 0 and I need the input at 1. So, that is what is this statements mean right.

Given an initial condition x_0 and given $u_0$to u(k-1), can I find x(k) and y(k). So, that is the reason I need information on this also right. Similarly to compute x_3 I need x_0, I need u at 0, u at 1 and u at 2 and generally for if I were to compute the value of state at any time k ok. As usual similar to the continuous time case, these equations will also have a natural response and a force response.

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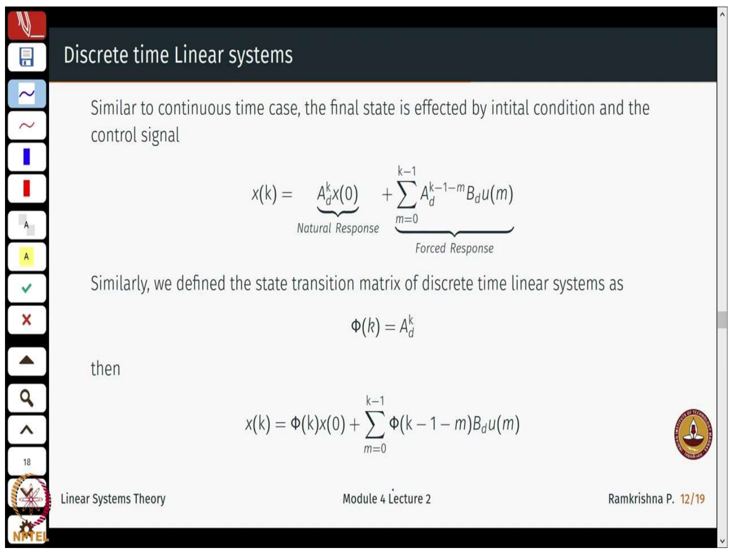

So, if I look at it closely, so if I just keep on doing this, and say for instance or for example, to be just to begin with assume u = 0 and left with an equation which is like this X at k is Ax_0. Well, this is suffix k just to denote it is a discrete time system x_0. And this A^k will now be my state transition matrix for the discrete time case and then I can write you know the written in general way as this form ok. Again the state transition matrix will be given certain initial condition what will my state be at say some arbitrary time k equal to 5 or 6 or whatever ok.

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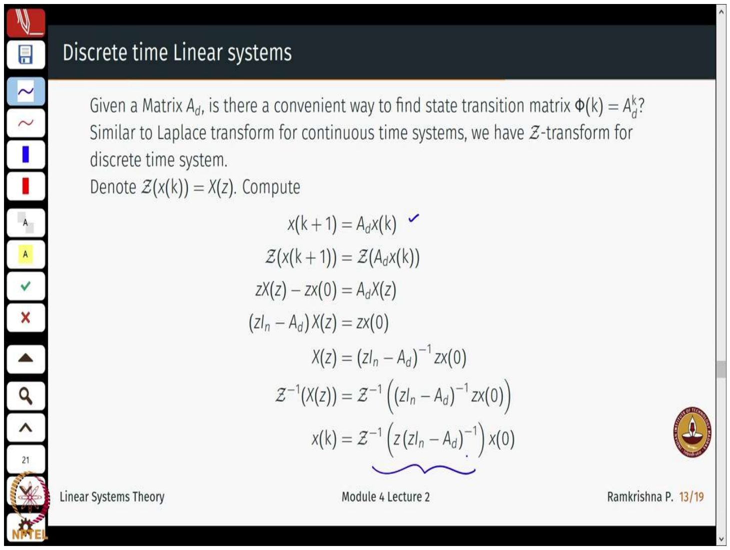

Now, again similar exercise how do we compute this matrix A^k, is it always easy to multiply the matrix n times. Well, similarly to what we have in the continuous time domain of Laplace transform, we have now the powerful tool called the z transform. And as usual I just start with a difference equation convert it into the z domain and then bring it back where the inverse z transform to my discrete time domain. Again the steps I will skip very similar to what we did in the case of the Laplace domain for solving continuous differential equations, ok.

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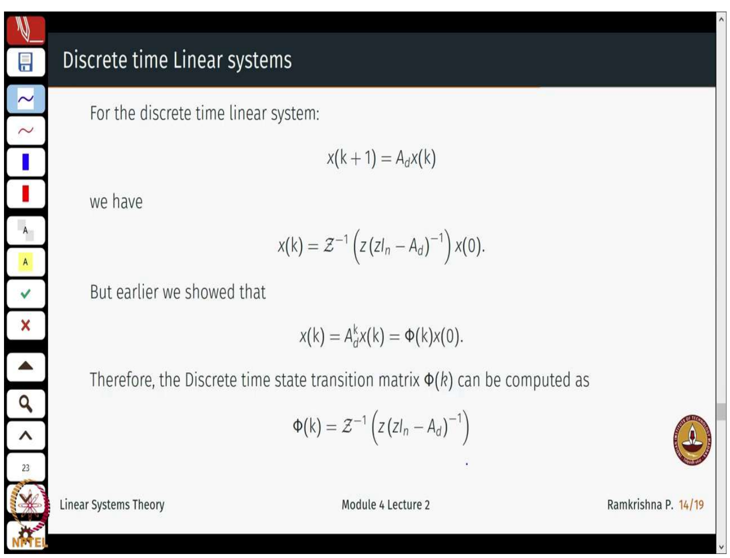

So, I will, so skip a bit of these steps, they are very ok. Just right down and check for yourself, they are pretty straightforward to verify.

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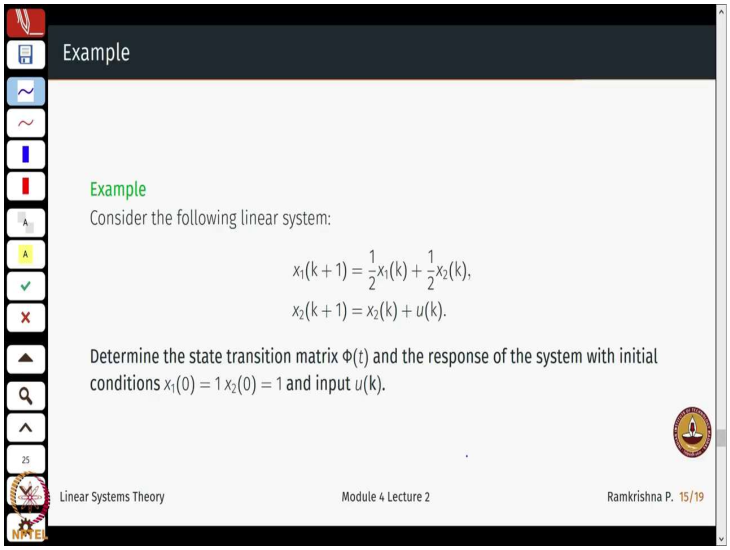

Even an example right; so, you can just work this example out for yourself it is.

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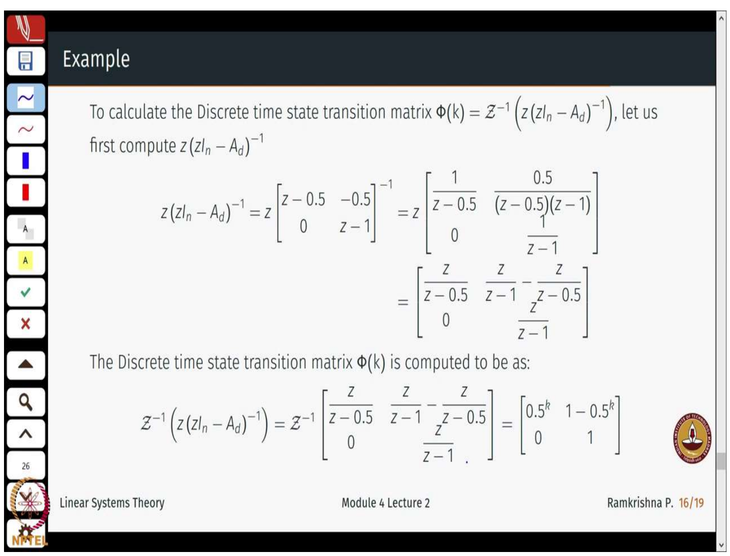

Again everything is neatly written down in the slides and you can just verify things for yourself, ok.

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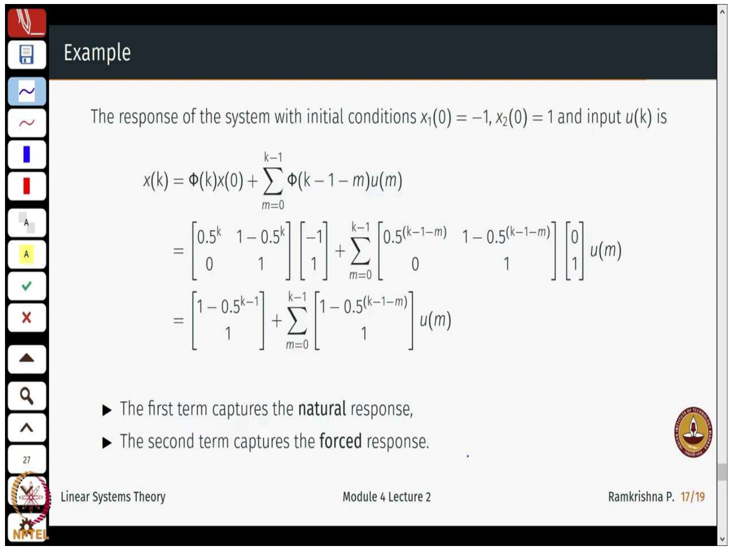

So, there are a couple of examples here which will help you understand how we compute it in discrete time. So, this will be a little useful in the later part of why we are actually doing discrete time systems. Much of the literature of today on networks science related concepts in control, they work with discrete time systems for obvious reasons which we will have. If we have time built a touch upon those kind of models a little later in the course when we have a decent amount of understanding of controllability and related stuff, ok.

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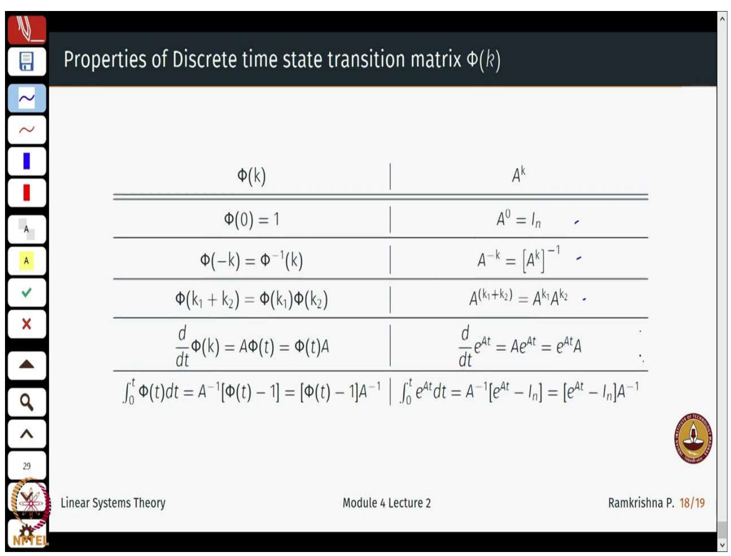

So, I will again leave this computations and again the kind of properties that we have here are again easy to verify of the trace state transition matrix. Very similar to what we had earlier, so this is easy to verify, this is easy to verify, this is easy to verify and so on ok; of course, this come from the continuous time case. So, we will end this lecture here with having learned how to compute force response of n-dimensional LTI systems, both for the continuous time and discrete time.

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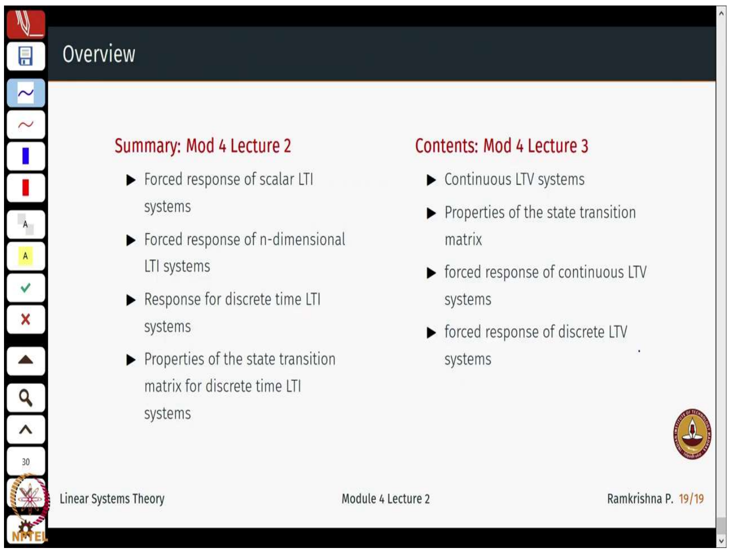

So, the next lecture we will focus on time varying systems, where we can also maybe find a little proof or why we started off finding an equivalence between the vector differential equation and the matrix differential equation. And why we had assumed that, so the solution to the matrix differential equation has direct implications on the solution of the vector differential equation $\dot{x}$ =Ax so, that we will do in the next lecture.

Thank you.