STOCHASTIC APPROXIMATION: THEORY AND APPLICATIONS

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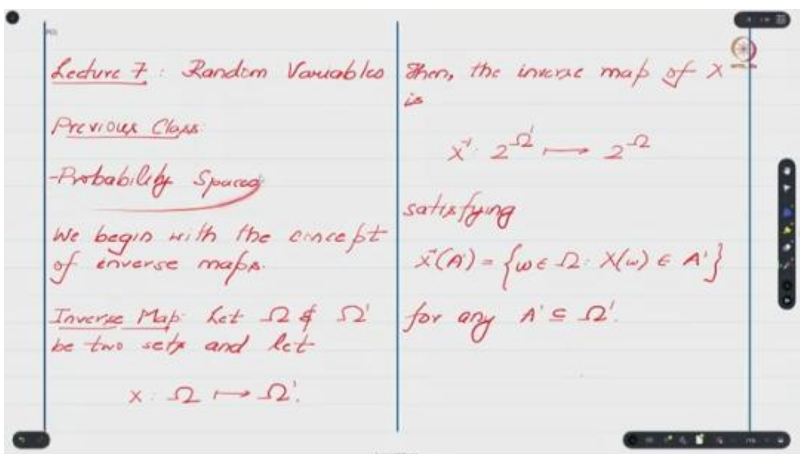
Lecture 7

Random Variables as Measurable Maps

Hello and Namaste, everyone. Welcome to Lecture 7. So, let us do a quick recap of what we did in Lecture 6. In Lecture 6, we looked at probability spaces, right? In particular, we understood what a probability space means.



In particular, we said that it is a triple made up of the sample space, the sigma field, and P. And in the last few minutes of the lecture, I also told you why, you know, the sigma field is necessary. The short answer is that, you know, if we could have defined the probability function on the power set always, then there would have been no need for working with the sigma field. But at the end, we saw an example where we could not do that, and hence we have to, you know, restrict ourselves to working with sigma fields. So, I would encourage you to think a bit about this and, you know, polish your understanding of the necessity of sigma fields and probability spaces and so on. So that the importance of sigma fields becomes clear to you, and our further discussions become smooth, right? And in this lecture, we will be looking at this concept called random variables.

In order to understand random variables, one of the things that we need to understand very closely is that of an inverse map. So, you must have heard of an inverse function. We are now going to define something in that spirit, but we are going to define it in a slightly generalized fashion so that certain things become easy to work with. So, what is an inverse map? So, let us say we have been given two sets, Omega and Omega prime, and let us say we have a function that goes from Omega to Omega prime.

Is this okay? Now, if you had to define an inverse function—and I am sure you all know that the inverse does not always exist—but whenever it exists, right, you would define the inverse function to go from omega prime to omega whenever it exists, right? However, the inverse map that we will define now does not go from omega prime to omega. Instead, it goes from the power set of omega prime to the power set of omega. Recall the power set is the collection of all subsets of that set.

So this will be the collection of all subsets of omega prime. Similarly, this would be the collection of all subsets of omega, and so on and so forth.

$$X: \Omega \mapsto \Omega'$$

$$X^{-1}: 2^{\Omega'} \mapsto 2^{\Omega}$$

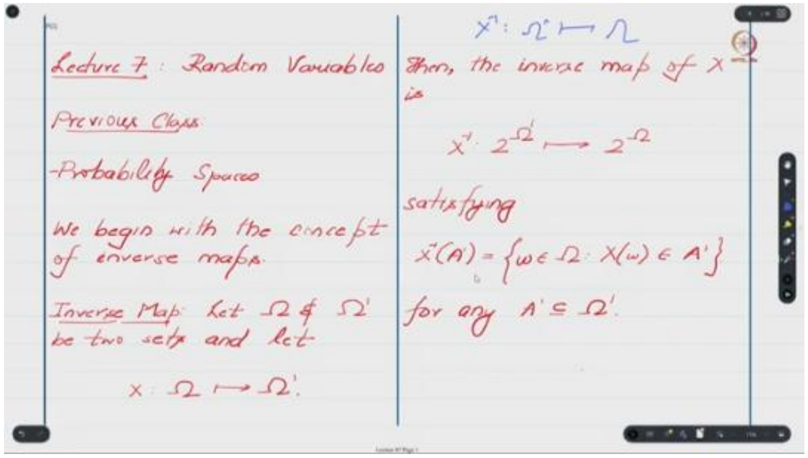
So, this inverse map takes as input a subset of omega prime and outputs an element which is a subset of omega. So, with this description in mind, what is X inverse of A prime? So, X inverse takes a subset of omega prime and maps it to this collection.

So, what is X inverse of A prime? X inverse of A prime is the collection of all little omegas in capital omega such that X of omega is in A prime. Is this okay? And this definition holds true for any subset of omega prime.

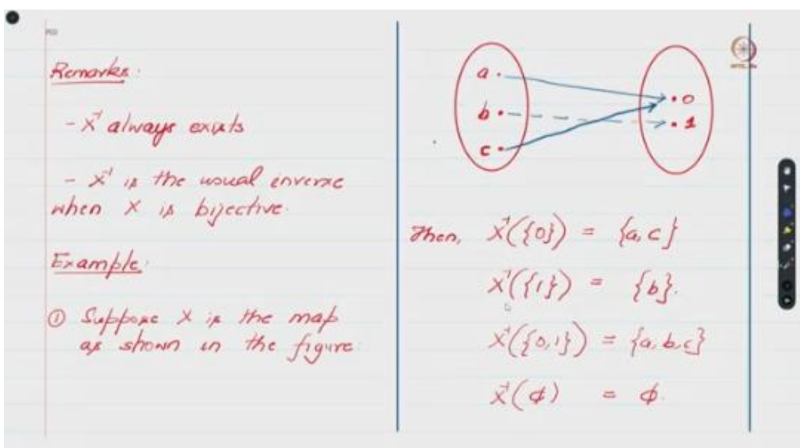
$$X^{-1}(A') = \{\omega \in \Omega: X(\omega) \in A'\}$$

$$A' \subseteq \Omega'$$

So, this is how we define the inverse map, right?



And the advantage of defining it this way is that the inverse map always exists, right? We do not have to worry about this one-to-one nature, this onto nature, and so on and so forth. You give me a map, and this inverse map will exist, right? No questions asked. Right, and then one can ask: Is there any relation between this inverse map and the inverse function you may have studied before? Well,

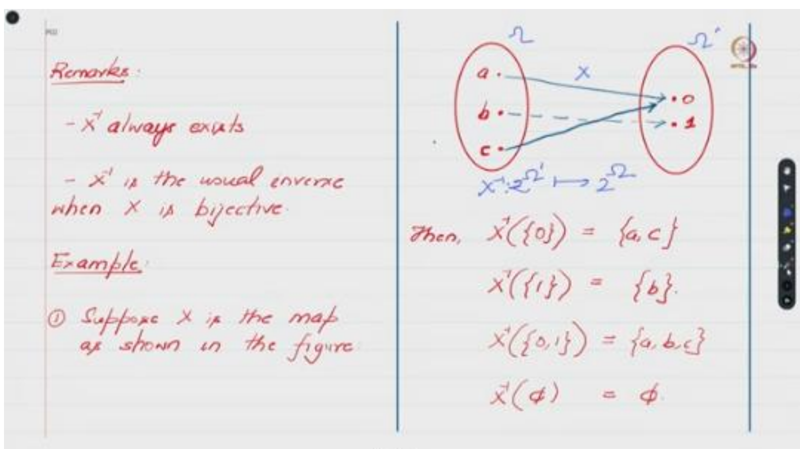


the answer is yes—when your given function is bijective, right? That is, one-to-one That is, one to one and onto, in that scenario, indeed, this X inverse you know has I mean is actually your usual inverse function, is this okay? So let us look at a quick example to understand this notion of an inverse map okay? So, let us say X is the map given in this example, right? So, in this case, you have your omega here and your omega prime here.

Omega is made up of these three elements: A, B, and C, whereas omega prime is made up of these two elements: 0 and 1, right? And this function X X maps A to 0, so function X maps A to 0, it maps C also to 0, and it maps B to 1. Is this okay? And I have just put this dotted line so that one can see which elements are mapped to 0 and which elements get mapped to 1. Other than that, the dotted line and the solid line have no other distinction.

So, one can clearly see that this function has a many-to-one nature; in particular, the elements A and C both get mapped to 0. Hence, in the traditional sense of the inverse function, there exists no inverse function here. However, we are claiming that the inverse map still exists. So, what is the inverse map? Well, the inverse map over here will go from the power set of omega prime and it will go to the power set of omega, right?

So, which means I have to tell you what is the output of X inverse when I give a subset of omega prime as input. So, what are the different possible subsets of omega prime? Well, it is 0, 1, the set containing both 0 and 1, and the empty set. And now, I have to figure out what is the inverse of this set 0. So, how do I figure out the inverse of 0?



I come to this omega set, identify all those elements which map to 0, and that would be the inverse map of this singleton set 0. Similarly, the inverse map of the singleton 1 would be all those elements in omega which map to 1. So, this is this set, and in that spirit, you can see that the inverse map of this set {0, 1} is basically {a, b, c}, and similarly, the inverse of the empty set is empty.

$$X^{-1} 2^{\Omega'} \mapsto 2^{\Omega}$$

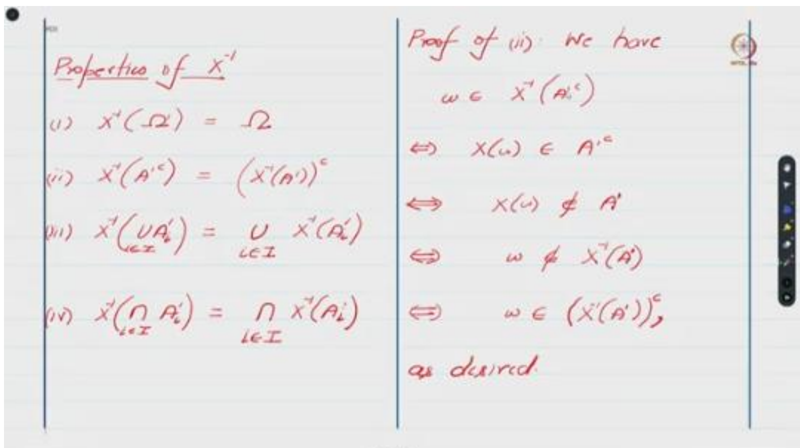
$$X^{-1}(\{0\}) = \{a, c\}$$

$$X^{-1}(\{1\}) = \{b\}$$

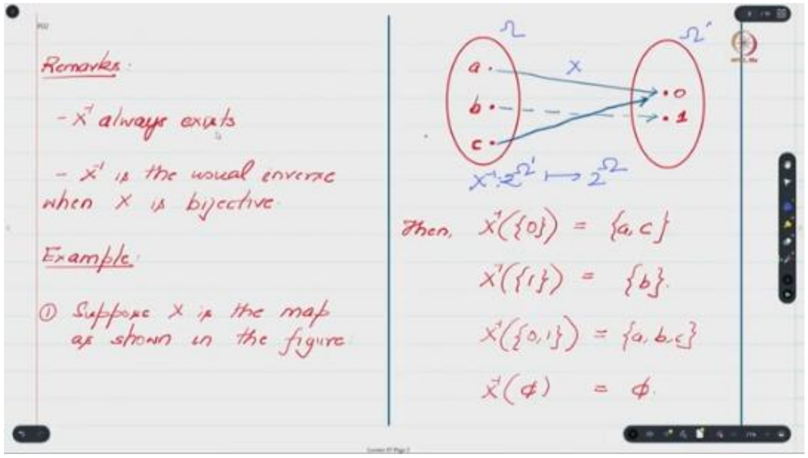
$$X^{-1}(\{0,1\}) = \{a, b, c\}$$

$$X^{-1}(\phi) = \phi$$

So, in that sense, you can see that your inverse map is defined even though this original function X is many-to-one. Now, this X inverse is something that plays a very, very important role in the study of probability and, in particular, random variables.



This X inverse, the way we have defined this inverse map, actually enjoys a rich set of properties which make it very powerful. I am going to list some of these properties, prove a few of them to give you a flavor of how one can go about proving these properties, and leave the rest as exercises for you to try out, right? So, the first of these properties is that if you give omega prime as input to this inverse map, then you will get back omega, okay? So, you can see in our previous slide that when I gave omega prime as input, you got back omega, okay? And the X inverse function, or the X inverse map, satisfies this property in general. That is, if you give omega prime as input, it will spit out omega.



Right? And it has some nice relation with the complement operation. In particular, let us say there was some A prime which was a subset of omega prime, and you looked at the complement. I denote this complement by this little c, and if you look at the inverse map of the complement of A prime, then it is the complement of the inverse map of A prime. That is, you first take the inverse map and then take this complement. Notice that this complement over here is with respect to the parent space omega, whereas this complement over here is defined with respect to the parent space omega prime.

But this X inverse, or this inverse map, has some nice structural properties which ensure that the complement operation and this inverse operation can actually be interchanged. And in the same way, one can see that the inverse operation of arbitrary unions. The index set here could be a finite set, it could be a countable set, or it could be an arbitrary uncountable set. So, whatever this index set is, if you take the inverse of this union, then it will be the union of the individual inverses. So, X inverse again has this nice property with respect to this arbitrary union operation, right?

And similarly, it has a nice relation with the arbitrary intersection property as well, right?

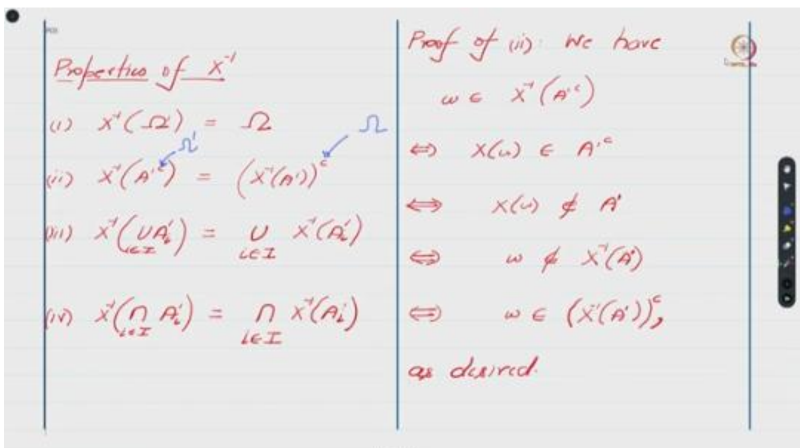
$$X^{-1}(\Omega') = \Omega$$

$$X^{-1}\left({A'}^{c}\right) = \left(X^{-1}(A')\right)^{c}$$

$$X^{-1}\left(\cup_{i\in I} A_i'\right) = \cup_{i\in I}\ X^{-1}(A_i')$$

$$X^{-1}(\cap_{i\in I} A_i') = \bigcap_{i\in I} X^{-1}(A_i')$$

And what I will do is, instead of trying to prove all these four properties that I have listed over here, I will prove this second property explicitly over here and then I will leave the proof of the other three properties as exercises, okay? So, let us see how we can go about proving it. So, recall that A prime is a subset of omega prime; in particular, A prime itself is a set. So, A prime complement is also a set, and X inverse of A prime complement is a set.



So, we are saying that this set equals this set. So, that is what we need to show. In order to show that, what we will do is let us pick some arbitrary element in X inverse of A prime complement. Now, from the definition of X inverse of A prime complement, it follows that omega belongs to this set if and only if X of omega belongs to A prime complement, right? This is from the definition of your X inverse.

Now, you know from the definition of complement, it follows that X of omega belongs to A prime complement if and only if X of omega does not belong to A prime, right. So, this slash over here over the belongs-to sign implies that X of omega does not belong to A prime. Now, X of omega does not belong to A prime, you know from the definition of X inverse, then leads us to conclude that this element omega does not belong to X inverse of A prime, right? And if omega does not belong to X inverse of A prime, we get that omega belongs to the complement of X inverse of A prime. And one can see that all these statements over here are if and only if, and hence one can conclude that omega belongs to

X inverse of A prime complement if and only if omega belongs to X inverse of A prime, the whole complement. And because of this if-and-only-if relation between them, this statement holds true.

$$\omega \in X^{-1}\left(A'^{c}\right)$$

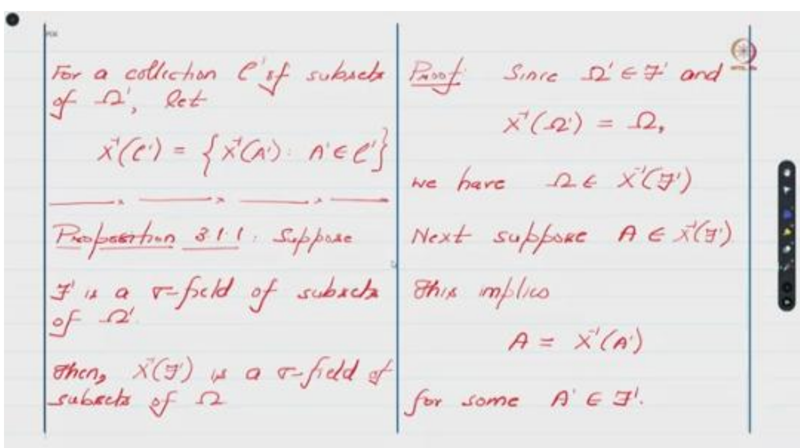
$$X(\omega) \in A'^{c}$$

$$X(\omega) \notin A'$$

$$\omega \notin X^{-1}(A')$$

$$\omega \in \left(X^{-1}(A')\right)^{c}$$

And in the same way, one can check the validity of these three statements as well. Now, one of the things that we will repeatedly require throughout subsequent discussions is this concept of the inverse of a collection of subsets. So, if you go back to the definition of X inverse, the input to X inverse should be one subset of omega prime. Now, what we are going to do is we are going to sort of overload the definition of X inverse in that instead of giving one subset as input to X inverse, we will take a collection of subsets of omega prime and give this whole collection as input to X inverse. Now, what will X inverse do?



Well, it will take the inverse of every subset that is there in C prime and put them all together. So, X inverse of C prime. So, C prime is a collection. So, X inverse of C prime is another collection, and this collection is made up of X inverse of A prime where A prime belongs to C prime.

$$X^{-1}(C') = \{X^{-1}(A') : A' \in C'\}$$

So, in this sense, we have defined or extended the definition of X inverse from one subset of omega prime to a collection of subsets of omega prime.

So, we have this definition over here, and this definition, once we have it, we will see that even under this generalization of X inverse from one subset of omega prime to a collection of subsets. There are several nice properties that are true for this definition. So, one of the key properties of X inverse is the following property. Suppose we have been given this sample space omega prime and we have been given a collection of subsets of omega prime, which can be viewed as a sigma field, right? So, F prime is a sigma field made up of subsets of omega prime.

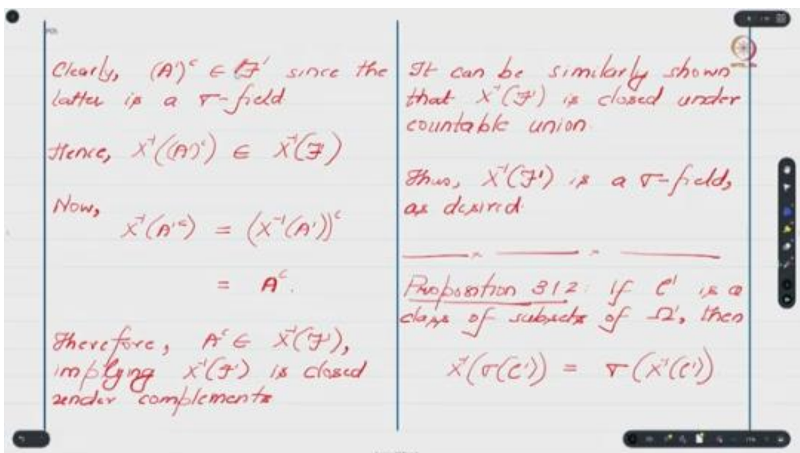
So, F prime is a collection of sets. And now we are looking at X inverse of F prime. So, this will be another collection of sets. However, this collection of sets will be the subsets of omega. And the claim is that if F prime is a sigma field, then X inverse of F prime is also a sigma field, but this time of the subsets of omega.

So, I will take you through the proof of this. So, it is very easy to prove this. You basically show that X inverse of F prime satisfies all the desired properties required by a sigma field, right? So, the first of these properties that we have to show is that omega belongs to this X inverse of F prime. The way we verify this is because your F prime is a sigma field made up of a collection of subsets of omega prime. The sample space omega prime itself should belong to F prime, and since omega prime belongs to F prime, X inverse of omega prime should belong to X inverse of F prime.

However, X inverse of omega prime is omega. Hence, one can conclude that omega belongs to X inverse of F prime. So, you know, this is a very easy sequence of arguments. Now, we move on and we show that this X inverse of F prime is actually closed under the complement operation. Is this okay?

So, what do I mean by that? I want to say that, let us say there is a set A in X inverse of F prime. So, X inverse of F prime is a collection, and let us suppose A belongs to X inverse of F prime. Our goal now is to show that A complement also lies in X inverse of F prime, right? So, let us go ahead and do that proof.

Since A belongs to X inverse of F prime, from the definition we have that there is some A prime in F prime such that A equals X inverse of A prime. So, from this definition, one can conclude that whenever A belongs to X inverse of F prime, there is A prime such that A has this relation over here. F prime, right, lies in F—sorry, A prime lies in F prime, right? That is what we have seen. And since F prime is a sigma field, A prime complement also lies in F prime, right?



Now, because A prime complement lies in F prime, X inverse of A prime complement lies in X inverse of F prime, right? And from this property of X inverse that we studied before, X inverse of A prime complement is basically the complement of X inverse of A prime, right? And X inverse of A prime was basically A, and hence the complement operation leads to A complement. Right. And because your X inverse of A prime complement lies in X inverse of F prime, one can conclude that A complement lies in X inverse of F prime.

And hence, one can now conclude that X inverse of F prime is actually closed under the complement operation. And one can similarly show that X inverse of F prime is closed under countable union, and from this, one can conclude that X inverse of F prime is indeed a sigma-field, as desired. So, what is the summary? If you have a collection of subsets of omega prime which forms a sigma-field, and if you take the inverse of this collection, then that new collection would actually be a sigma-field of subsets of omega. Now, one can go ahead and extend this property to ask: what if the sigma-field of subsets of omega prime was generated by this collection C prime, right?

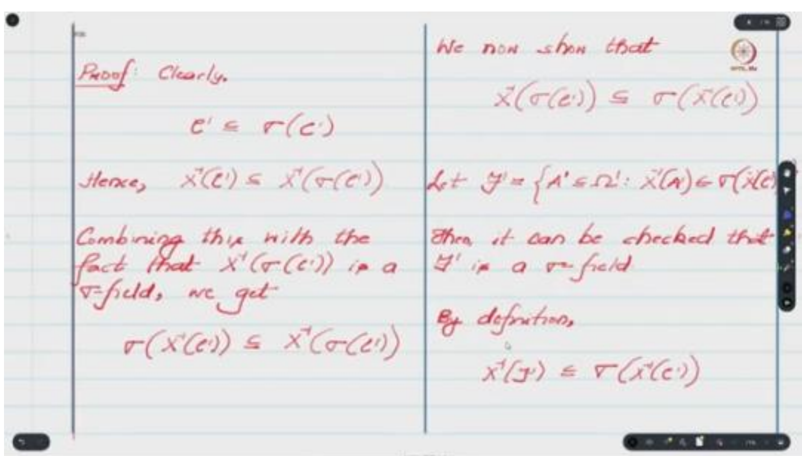
So, can we say something about the property under this X inverse? So, this very nice property actually holds in that case. So, let us try to understand this property. So, you take C prime. Look at the sigma-field that is generated by C prime.

This is sigma of C prime. You take its X inverse. So, we know from the previous result that this would be a sigma-field because sigma of C prime is a sigma-field. Hence, X inverse of this will be another sigma-field but of the subsets in omega. Now, on the other hand, suppose you first took the inverse of C prime itself.

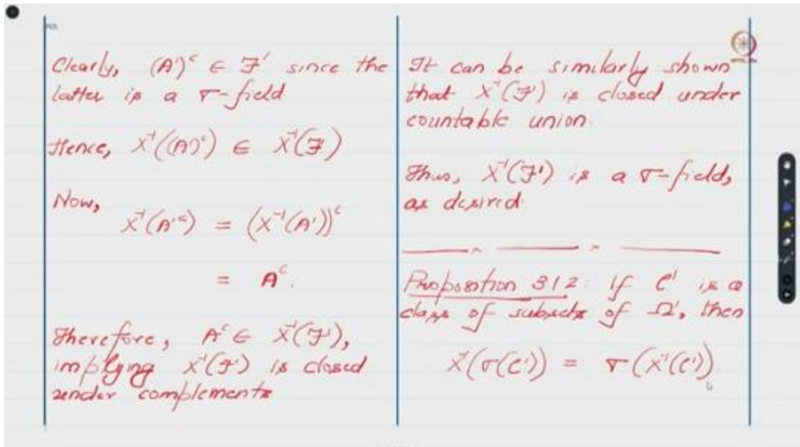
So, C prime is a collection. You take X inverse, that will be another collection. Since C prime has not been defined as a sigma field, X inverse of C prime also need not be a sigma field. So, now you look at the smallest sigma field containing this X inverse C prime, and one can show that these two quantities are one and the same.

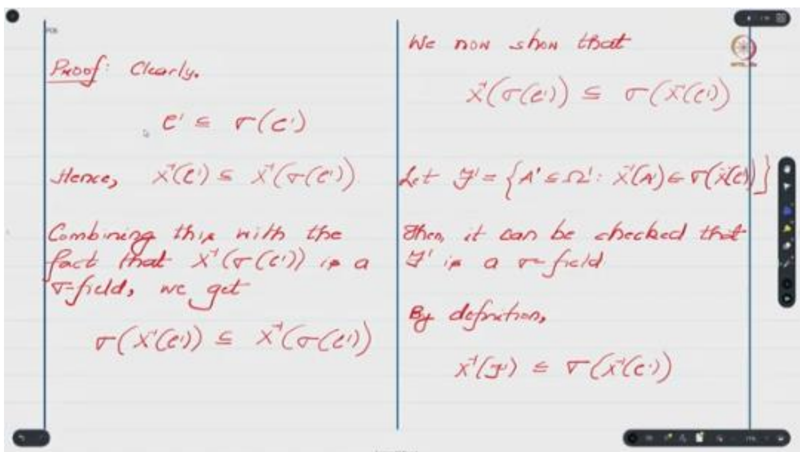
$$X^{-1}(\sigma(C')) = \sigma(X^{-1}(C'))$$

So, let us have a quick look at the proof.



So, this is a collection, this is a collection of subsets of omega, and we are going to show that this collection and this collection are the same. One way to show that these two collections are the same is to show that each of these collections contains the other. So, this contains this, and similarly, this collection contains this. So, we will actually prove it in that fashion. So, the proof goes as follows. You know, we have been given some collection.





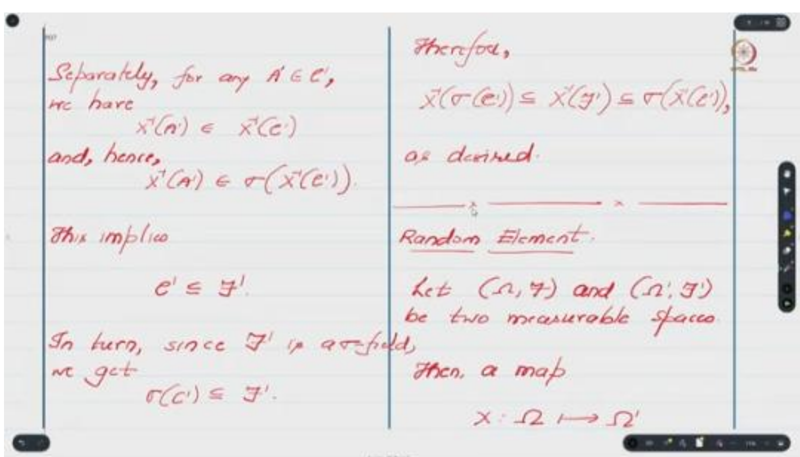
This is the smallest sigma field containing C prime. Hence, trivially it follows that C prime is a subset of sigma of C prime, right? And from this property, one can see that X inverse of C prime is a subset of X inverse of sigma of C prime. Is this okay? Now, from the previous result, we know that if sigma of C prime is a sigma field, then X inverse of sigma of C prime is also a sigma field, and this sigma field consists of X inverse of C prime.

And hence, if we look at the smallest sigma field containing X inverse of C prime, it must be a subset of this. This is from the property of sigma of X inverse of C prime being the smallest sigma field containing X inverse of C prime. So, from this set of arguments, one can conclude that this containment relation holds, that is, The smallest sigma field containing X inverse of C prime is actually a subset of the inverse image of the smallest sigma field containing C prime. So, we have shown one containment relation, and now we will show the other containment relation. This proof actually progresses by a very clever argument, which is given in Resnick's textbook, and I will follow that clever argument.

So, what this clever argument does is, it first looks at a collection of subsets of omega prime which satisfies the following property. So, what does it say? F prime consists of all those subsets of omega prime such that X inverse of A prime belongs to the smallest sigma field generated by X inverse of C prime, okay? So, it is made up of all A prime such that X inverse of A prime lies in the smallest sigma field generated by X inverse of C prime. I would leave it as an exercise to show that this collection is actually a sigma field. Now, assuming that this F prime, defined over here, is a sigma field, we will try to establish this relation that we have stated over here.

So, by definition, F prime consists of elements of this form, right? And because of this relation, one can see that X inverse of F prime, which will basically contain X inverse of A prime for every A prime in F prime, right? So, this will be a subset of what we have over here. So, this property one can trivially check, just holds from the definition of F prime. Is this okay?

Now, separately, if you take any A prime in C prime, right? If you take any A prime in C prime, right? It is trivially true that X inverse of A prime, right? So, the inverse map applied to this set A prime will lie in the collection X inverse of C prime. So, C prime is a collection.



If you take X inverse of C prime, that will be another collection, and it will be made up of X inverse of A prime for every A prime in C prime. Hence, from that definition of X inverse of C prime, it follows that X inverse of A prime lies in X inverse of C prime. And from this property, one can conclude that X inverse of A prime must be in the smallest sigma field containing X inverse of C prime. So, if X inverse of A prime lies here, then trivially, X inverse of A prime must be in the smallest sigma field containing X inverse of C prime. So, from this, one can conclude that C prime is a subset of F prime.

So why do I say that? Because F prime consists of all those A primes which satisfy this relation, and what we have shown here is that for every A prime in C prime, this relationship is true, and hence it must follow that C prime is a subset of F prime. Is this okay? Now, because F prime is a sigma field, right, and it contains C prime, it follows that the smallest sigma field containing C prime is actually a subset of F prime. Now, because of this relation, it follows that X inverse of sigma of C prime is a subset of X inverse of F prime.

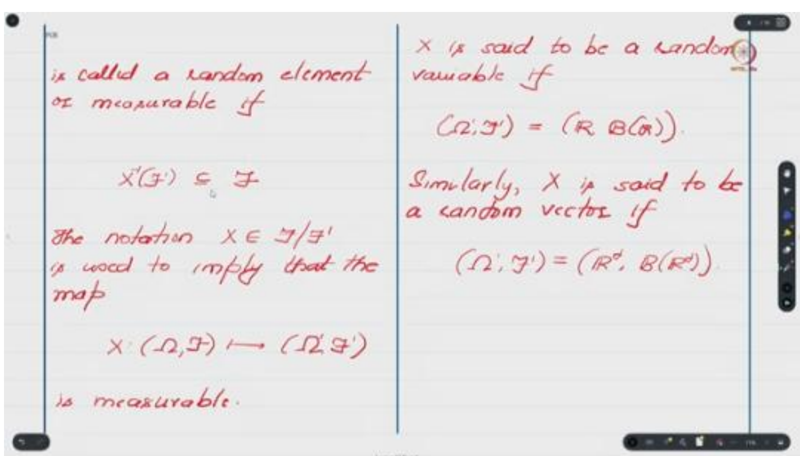
And in our previous discussion, we had shown that X inverse of F prime is a subset of sigma of X inverse of C prime, and that is what we have written over here. And from this, one can conclude that X inverse of sigma of C prime is a subset of this, as we had desired to show. Is this okay? So, let me now summarize what we have done so far. We have defined this inverse map.

First, we defined this inverse map on the power set of omega prime. Then, we extended this definition of the inverse map from the power set to a collection of subsets of omega prime. And then, we showed that this X inverse map actually enjoys several interesting properties. In particular, we showed that if you have a sigma field of subsets of omega prime, it can be taken back to the sigma field of subsets of omega. And then, we said that if you have a collection C prime of subsets of omega prime and if you look at the smallest sigma field generated by the C prime, then you know it does not matter whether you first take the inverse of C prime and then look at the smallest sigma field containing that inverse. Or you first take the smallest sigma field containing the C prime and then take the inverse.

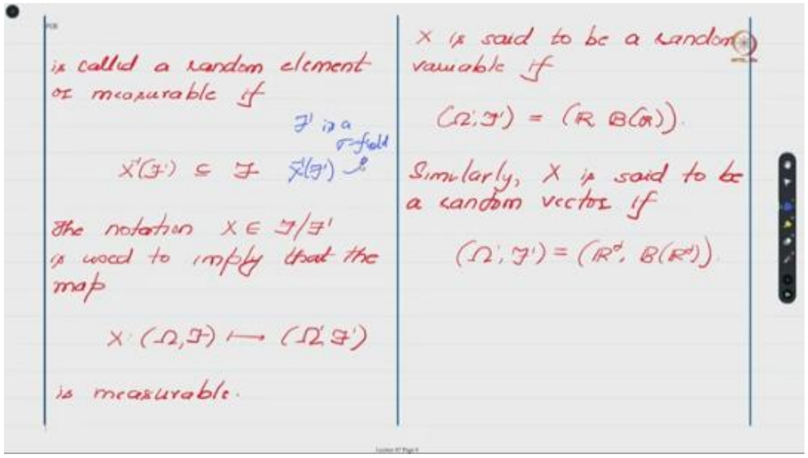
And we showed that both these things are one and the same. This is actually a very nice property of X inverse and shows that this X inverse is a very nice definition and allows us to do a lot of algebra in the context of probability in a very elegant fashion. So, having given this background, we will now move on to defining what is called a random variable, and towards that, we will first define what is called a random element. So, let us suppose that we are given two measurable spaces. Okay, so what are these two measurable spaces? The first of these measurable spaces is omega comma F, and the second of these measurable spaces is omega prime comma F prime, okay?

So, what do i mean by a measurable space a measurable space is basically a tuple made up of a set and a sigma field, right? So, the first measurable space is made up of a set and a sigma-field. The second measurable space is made up of another set and another sigma-field, right? And a measurable space is basically a space on which we can define a measure. And since we are talking about probability, the idea here is that this is the space on which we can define the probability measure.

Is this okay? So, we will come to that later. So, at this point, we want to define a random element. What is a random element? Well, a random element is something that we will denote by the letter X, which is a function that will go from Ω to Ω', and it satisfies the following property: that is, $X^{-1}(F')$ is a subset of F. So, let us understand this

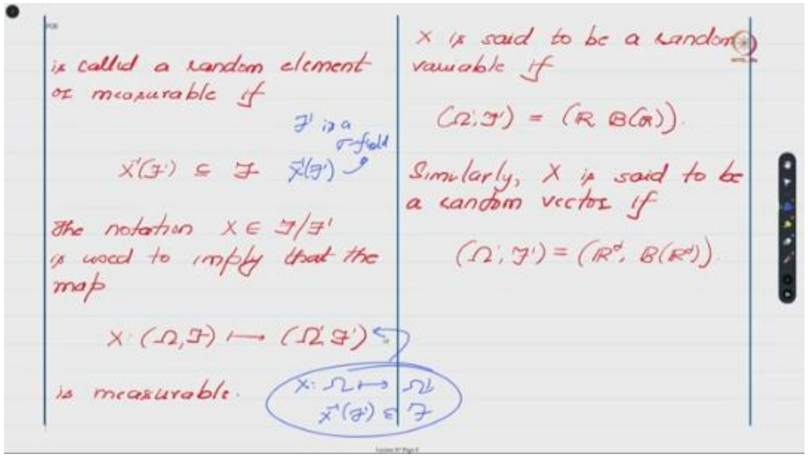


a bit more. Your F′ is a sigma-field, right? And we know that X inverse of F prime is also a sigma field. This also is a sigma field. Of course, F prime is a sigma field of subsets of omega prime, whereas X inverse of F prime is a sigma field of subsets of omega.

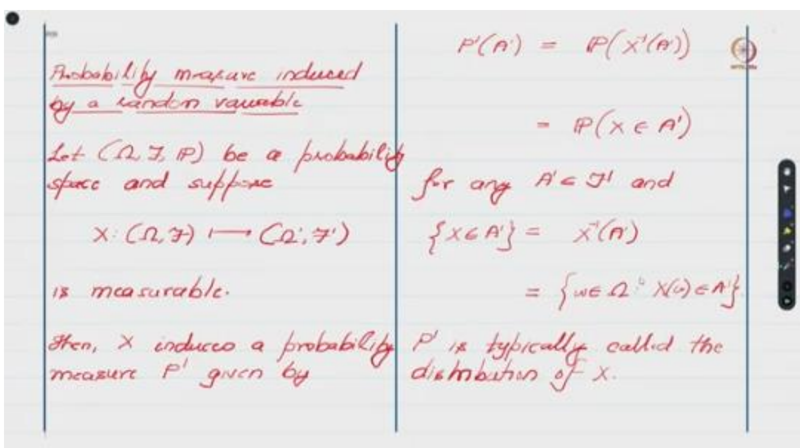


And we will say this X is a measurable map with respect to these two sigma fields, that is, calligraphic F and calligraphic F prime, if this condition actually holds. So, in general, this need not hold, but whenever such a condition holds, we will say that X is actually a measurable map—that is, it respects these two sigma fields that we have in place. And whenever we want to say X is measurable with respect to that space, we will, you know, use this shorthand notation. Right, or alternatively, write it in this fashion. So, one has to be careful over here. When I write it in this fashion, I don't mean that X actually is a map between Omega F and Omega prime F prime. Instead, X is a map from Omega to Omega prime, and it satisfies the property that X inverse of F prime is a subset of—sorry—is a subset of. So, these two things together we will compactly write in this fashion.

Now, one can ask: we know what a random element is—is it related to a random variable? Well, of course, it is related to a random variable. A random variable is a special instance of a random element. Where your omega prime F prime—that is, the second measurable space—equals this collection or this tuple, right? The omega prime set is the set of real numbers, and your F prime—that is, the sigma field defined on omega prime—is basically the Borel sigma field defined on R.



Is this okay? And similarly, we can define what a random vector is. Well, a random vector is a random element where omega prime F prime is R^d and the Borel sigma field defined on R^d, and so on. So, when you have such a definition, you know, one can extend this idea to define more complicated objects, such as, you know, random processes and so on and so forth. So, now that we understand what a random variable is, we will just, you know, quickly understand this concept called the probability measure that is induced by a random variable.



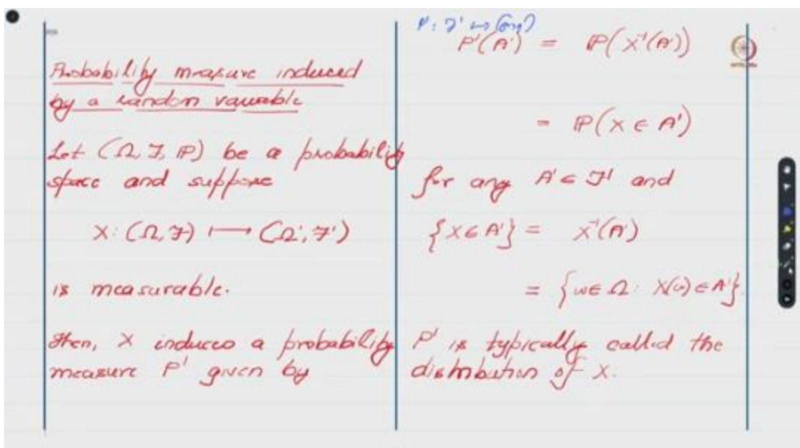
So, you will soon see that often we will talk about the distribution associated with a random variable, and when we talk about that, at that point in time, this concept will actually be very useful. So, what are we talking about? Well, we will say that, let us say, we have been given some probability space. So, we have a measurable space on top of that, some probability measure has been defined, and in addition, we have been given a random element which goes from this measurable space to this measurable space.

$$X: (\Omega, \mathcal{F}) \mapsto (\Omega', \mathcal{F}')$$

And the claim is that whenever you have such a random variable, it actually induces a probability measure, a new measure, on this measurable space.

Is this okay? So, if you want to define this new measure P prime, recall that this has to go from F prime to 0, 1. to 0, 1, and it should satisfy certain properties. So, what are the properties that it should satisfy? Well, it should assign the value of 1 to omega prime, and it should have this sigma-additive property.

And how is this P prime defined? Well, this P prime is defined in the following way. Suppose somebody gives you some A prime in F prime, right? What you do is, you first look at X inverse of A prime. So, A prime is a subset of omega prime.



You take X inverse of A prime, that will be a subset of omega, and because this X is measurable, this X inverse of A prime will be an element in F. Okay, because this is an element in F, and because we have been, you know, given this probability measure P, right? For every element in F, P assigns some value, right. So, we can talk about this thing, and P prime of A prime is basically P of X inverse of A prime. And we will often denote this X inverse of A prime as, in this way, X belongs to A prime. Hence, we will say that P prime of A prime is P of X belongs to A prime.

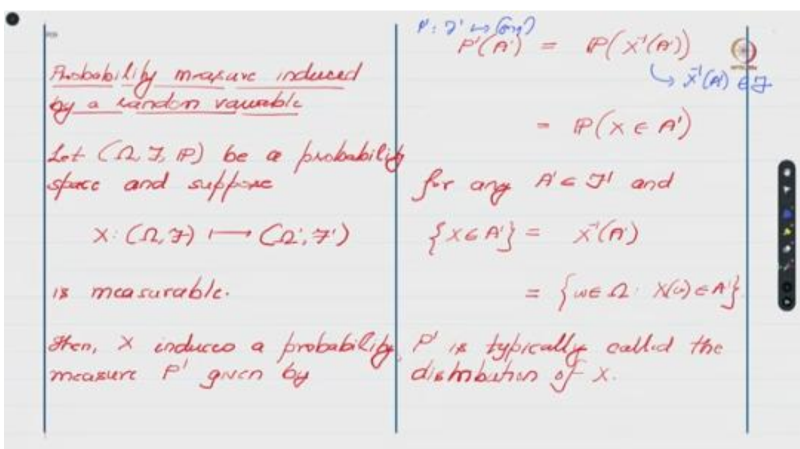
Now, notice that this right-hand side is defined or given to us via the definition of P. And we are using this definition to come up with a new definition, P prime on F prime. Is this okay? That is why we say that a random variable X is extending this probability measure that is defined on the original space. to this new measurable space omega prime F prime.

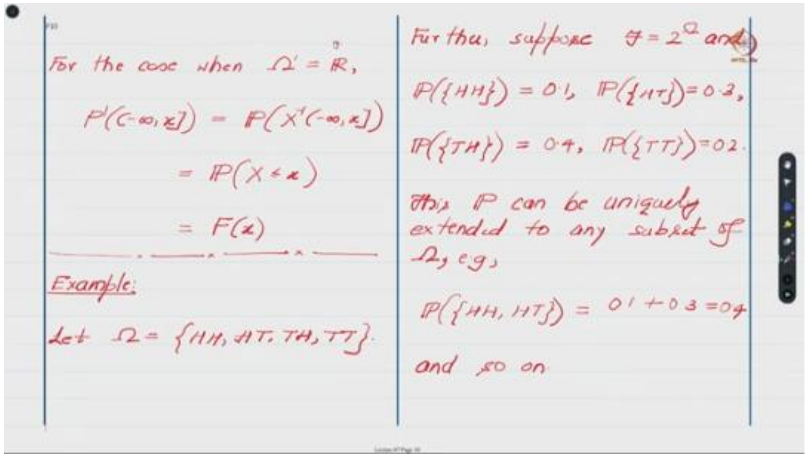
Is this okay?

$$P'(A') = \mathbb{P}\big(X^{-1}(A')\big)$$

$$= \mathbb{P}(X \in A')$$

And we will refer to this P prime often as the distribution of the random variable X. Is this okay? And you know, when this omega prime is R, one can see that P prime of this particular set, which is the interval minus infinity comma x, this will be the subset of R and this is an element in the Borel sigma-field on R. So, one can ask what is this P prime of minus infinity comma x? Well, this is the probability that is assigned to X inverse of minus infinity comma X, and this is formally written—or informally written, the way you want to look at it—as P of X less than or equal to X, and this is often denoted by this common quantity called the cumulative distribution function, right? The CDF.





So, this is the CDF associated with X. So, P prime of this thing is basically the CDF, and the way this is defined is in terms of the original probability measure assigned on this set or subset of omega. So, let us look at an example. So, let us look at this example of tossing two coins. In that case, your omega will be made up of these four outcomes: HH, HT, TH, and TT.

And because this omega is a discrete set—in particular, finite—we can actually work with a sigma field, which is a power set. Is this okay? And what we will do is, you know, we will first define a probability measure on omega. In particular, we will define it on this sigma field, right? And it is defined in this way.

For the singleton—you know, event or the—sorry, I should not say singleton event. So, for the singleton set HH, let us say P assigns the value 0.1; it assigns 0.3 to the singleton HT, and so on and so forth, right? Now, you know, we have ensured that all these values are non-negative and they add up to 1. Is this okay? Now, the way we described how we can extend P in the previous class, one can show that once you have defined P for the singleton elements, one can extend P to any subset of omega using, you know, this simple formula: when you have two elements, you add the values that you assign to the singleton, and so on.

And we do it this way so that P respects the sigma-additive property, okay? And by, you know, extending P to any subset of omega, one has now—or we have now managed to—define P on the power set of omega, which is a sigma-field, okay. So, we now have a measurable space—that is, sorry, a probability space—that is, we have sigma, We have

2^sigma, and we have P, where P goes from 2^sigma to [0, 1], right? Which satisfies all the desired properties, which is that P of omega should be 1 and P should be sigma-additive.

$$\Omega = \{HH, HT, TH, TT\}$$

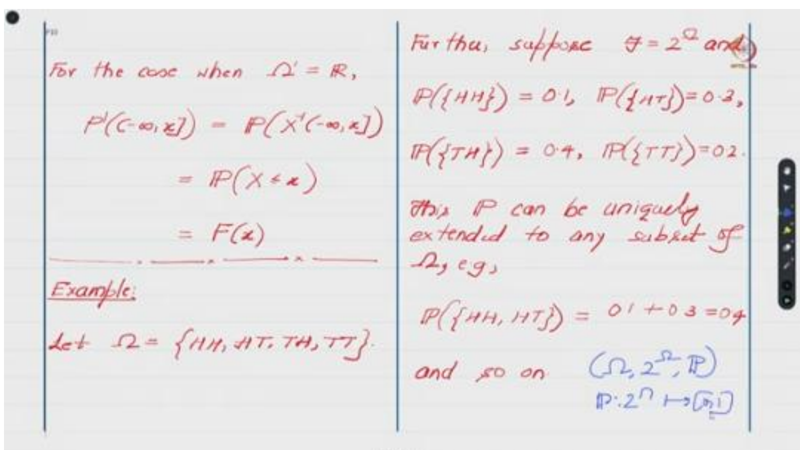
$$\mathcal{F} = 2^{\Omega}$$

$$\mathbb{P}(\{HH\}) = 0.1, \mathbb{P}(\{HT\}) = 0.3$$

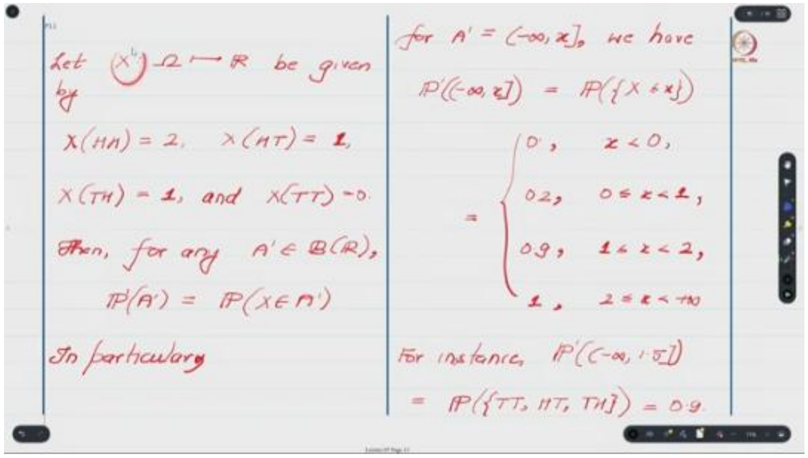
$$\mathbb{P}(\{TH\}) = 0.4, \mathbb{P}(\{TT\}) = 0.2$$

$$\mathbb{P}(\{HH, HT\}) = 0.1 + 0.3 = 0.4$$

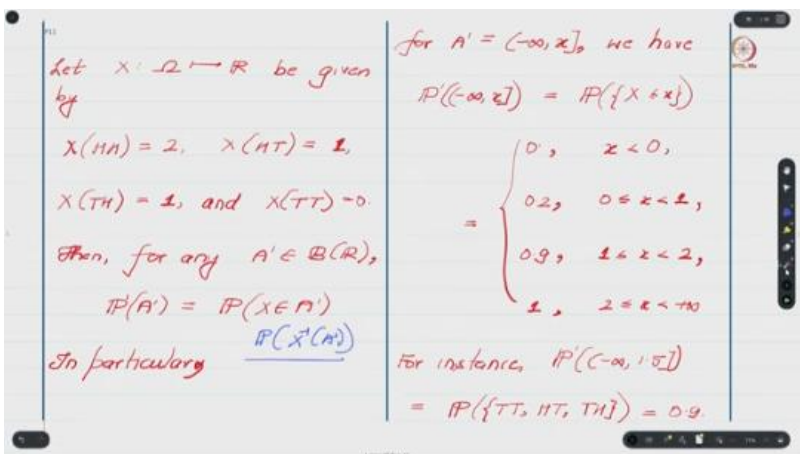
Is this okay? All right.



So, now let us define a random variable X, which goes from capital omega to R, and let us say, you know, it counts the number of heads. Is this okay? So, on HH, X will take the value 2; on HT, it will take the value 1; on TH, it will take the value 1; and on TT, it will take the value 0. So, now one can ask, given this random variable X, what is the distribution that X induces on R? Is this okay?



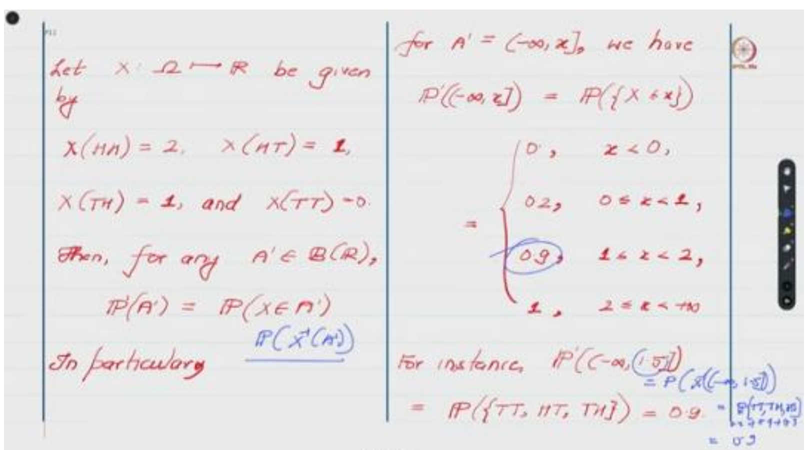
And we had denoted this induced distribution by P prime. And we had defined P prime of A prime to be the probability associated with X belonging to the A prime set. So, recall that this is basically P of X inverse of A prime, right? So, using this definition, one can now check what is the distribution that is induced on R. And by that, I mean what value P prime associates to every set that is there in the Borel sigma-field of R.



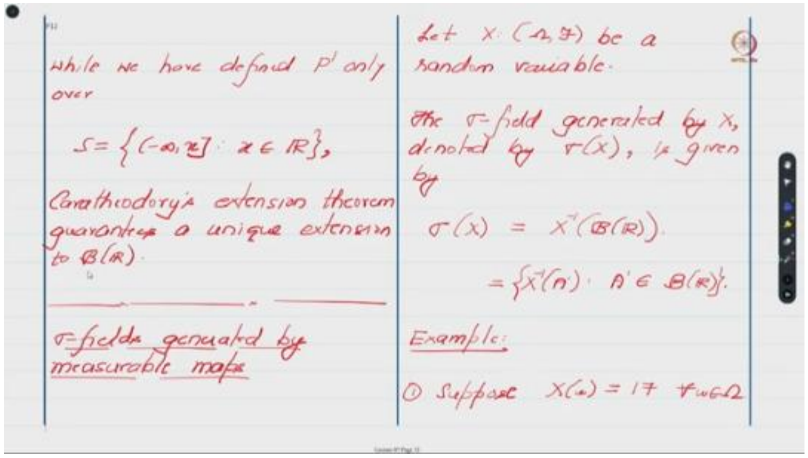
So for that, what we will do is we will first define this P prime on sets of this form. Right, and one can very quickly see that, based on the probability values that we had assigned before, that P prime of minus infinity comma x will take these four values for the different ranges. When x is less than 0, you know, P prime will assign the value 0. When x is between 0 and 1, P prime will assign the value minus—sorry, P prime will assign the value of 0.2 for, you know, this interval, and so on and so forth, right? And one can, you know, quickly check why that is the case.

So, if you take P prime of the interval minus infinity to 1.5, right? So, what we have to do is we have to look for this equals P of X inverse of minus infinity comma 1.5, right? So, we have to look at this, and this is basically all those elements where x assigns the value between minus infinity and 1, and this is basically P of the set, you know, tt, th, and ht. So, if you look at tt, th, and ht, and if you go back, so tt was assigned the value 0.2, 0.2 plus TH was 0.4, 0.4, and HT was assigned the value 0.3. So, if I add 0.3 over here, you will see that this turns out to be 0.3.

Is this okay? And from this, one can see that this is how you get the value of 0.9 when X equals 1.5, and you can similarly calculate the other values, okay. So, we wanted to define this P prime, okay, on the Borel sigma field. So, what we have done is we have first defined it for sets of this form. And in our previous discussion, we had seen that this collection of sets of this form actually forms a semi-algebra. And once we have a semi-algebra, we can extend it by using the Carathéodory extension theorem to a probability measure on this, you know, Borel field.

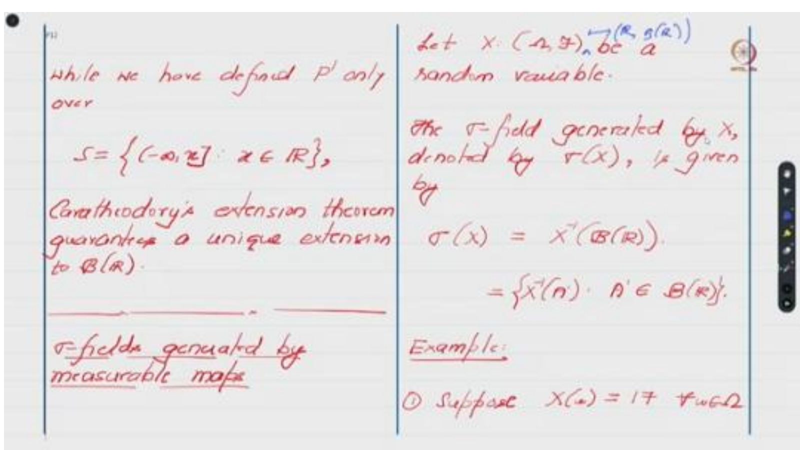


Is this okay? So, in the last couple of minutes, I am going to talk about sigma fields that are generated by these random elements. So, we are going to talk about that. So, this is again one important concept that we will talk about, particularly when we define conditional expectations. We will define conditional expectations with respect to such sigma fields. So, you will soon see how we make use of the sigma field.



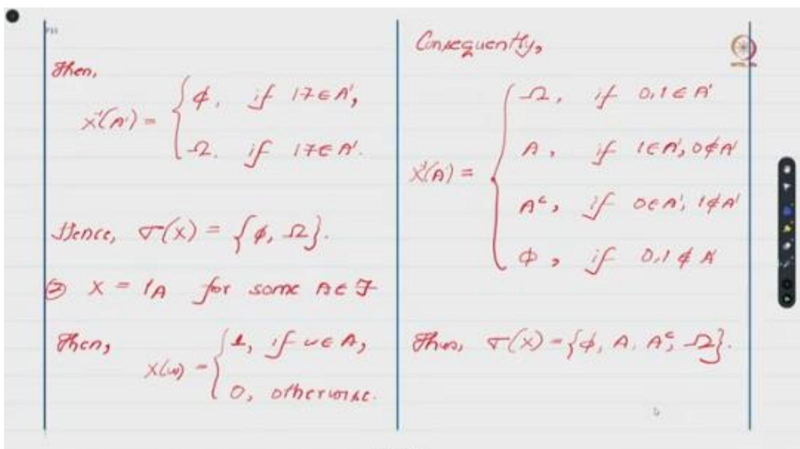
So, let us first look at the definition of such a sigma field. So, let us go back to this discussion. So, let X be a random variable. So, I think I have left out some description here. So, I should say, let X from omega F to R comma BR be a random variable.

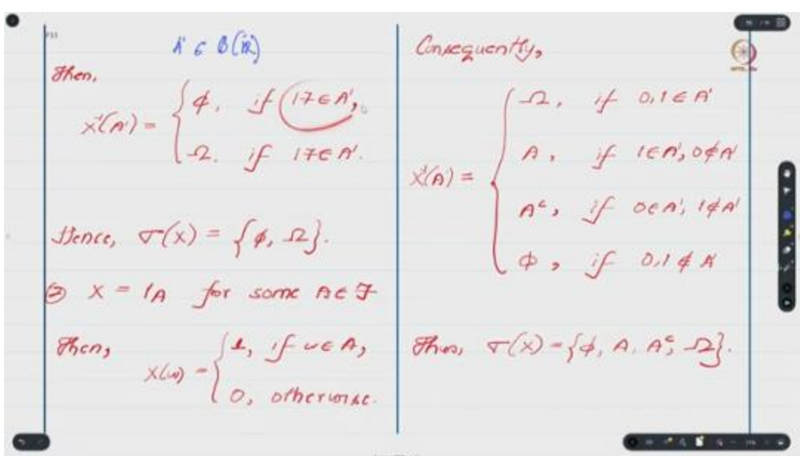
Now, the question we are asking is: what is the sigma field generated by X? This question should appear a bit strange to you because, so far, we have looked at sigma fields generated by a collection of subsets. Now, what is this sigma field generated by a random variable? Well, this sigma field generated by a random variable, we will denote it by sigma of X, and it is defined in the following way. So, we take the Borel sigma field of R. Now, you know that this itself is a sigma field. So, you take the inverse of that.



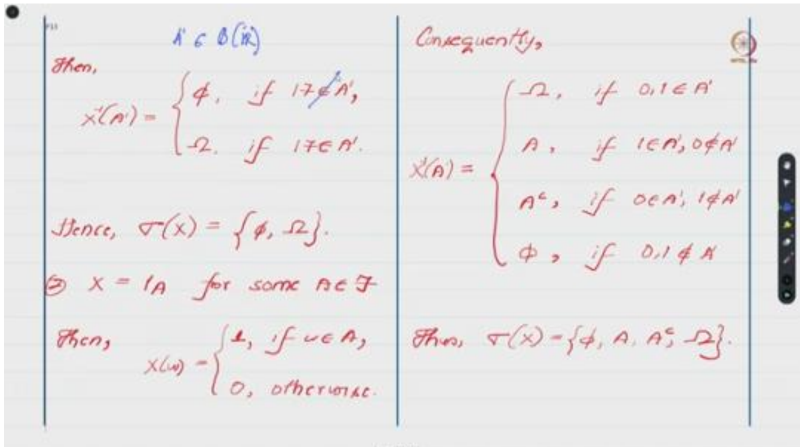
Now, we know that if we have a sigma field and you take its inverse, that itself will be a sigma field. So, this will be the sigma field of subsets of omega, and you define sigma of X to be this collection over here. Is this okay? So, this is the sigma field that is generated by this random variable X, and one can equivalently see that it is the inverse of A prime for every A prime in your Borel sigma field. So, let us look at a couple of examples.

You know you have a very trivial map which takes every element little omega to capital omega and maps it, let us say, to the value 17. So now one can ask what is the sigma field generated by this very, very trivial map, and the answer is the sigma field generated by this is basically made up of the empty set and the whole space. So let us see why that is. So, what I have to do is I have to basically look at, you know, A prime in X inverse of—sorry, I have to look at A prime in your Borel sigma field. Okay, and then I have to look at X inverse of A prime, right? Now, whatever be this Borel sigma—I mean, whatever be this A prime—either 17 belongs to A prime or 17 does not belong to A prime, okay.





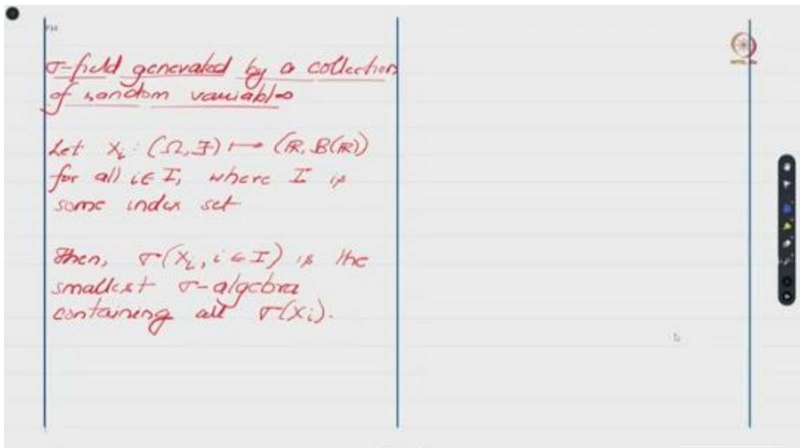
So, I have made a typo over here. So, either 17 belongs to A prime or 17 does not belong to A prime. If 17 belongs to A prime, I think I made a mistake; it should be 17 does not belong to A prime, right? So, if 17 does not belong to A prime, X inverse of A prime, which basically includes all those elements in omega, right? You know, which map to A prime, but you know this X is very special; it maps everything to 17.



So if A prime does not include 17, right? Then none of the elements in capital omega will map to this A prime. Hence, X inverse of A prime will be this empty set. On the other hand, if 17 belongs to A prime, then one can trivially check that X inverse of A prime is actually the whole sample space omega. And because X inverse of A prime can either be the empty set or this whole sample space, one can conclude that sigma of X basically includes this. So, here is another example. Let us say X is the indicator random variable associated with this set A.

So, let A be a set in this given sigma field F, and let I of 1 subscript A denote the indicator, which means that X of omega is 1 whenever omega belongs to A and it is 0 otherwise. In this case, one can see very easily that X inverse of A prime is satisfying these four values depending on whether 0 and 1 belong to A prime, or 0 does not belong to A prime, 1 belongs to A prime, and so on and so forth. You can very easily verify that, and because of this verification, one can see that the sigma field generated by X consists of the empty set, A, A complement, and omega. Is this okay? Alright.

So, I now come to the last concept of today's class, which is the sigma field that is generated by a collection of random variables, okay? So, in the previous two slides, we looked at the sigma field generated by one random variable. In this last slide, we are going to talk about the sigma field that is generated by a collection of random variables, okay? So, here we are going to look at a bunch of random variables x1, x2, x3, and so on—more generally, xi for little i in some index set I, right? And now we want to look at the sigma field generated by this collection. Well, the definition is that it is the smallest sigma algebra which contains all the sigma algebras that are generated by the different random variables.



So, sigma of xi is the sigma-field generated by xi. You collect all these sigma of xi's, and now this is a collection of subsets of omega. You put all of them together and ask: what is the sigma-algebra, or the smallest sigma-algebra containing this whole collection? Is this okay? And that collection will be referred to as the sigma-algebra generated by the collection of random variables xi, where i belongs to capital I.

So, this brings me to the end of this class. Let me quickly summarize what we did. Again, in this class, we have covered a lot of material so that we can go over the background material very quickly. So, we can focus the future lectures on our main topic, which is that of conditional expectation, martingales, and so on and so forth. So, in this class, we focused on understanding this concept called the inverse map.

And using that, we defined this concept called a random element. Then we said that a random variable, which all of you must be familiar with, is a special instance of a random element where this omega prime f prime is your R and the Borel sigma-algebra defined on R. We looked at the probability measure that is induced by a random variable X, right? And eventually, we talked about the sigma-field that is generated by a random variable, and we then extended that definition to the sigma-field that is generated by a collection of random variables. In the next couple of classes, we will first look at the expectation of a random variable, and then we will look at the notion of conditional expectations. And once we are familiar with these notations and so on and so forth, we will move on to look at what are called martingale differences. In particular, we will look at the convergence of martingale difference sequences.

With this, let me end today's class. Thank you. See you in the next lecture.