STOCHASTIC APPROXIMATION: THEORY AND APPLICATIONS

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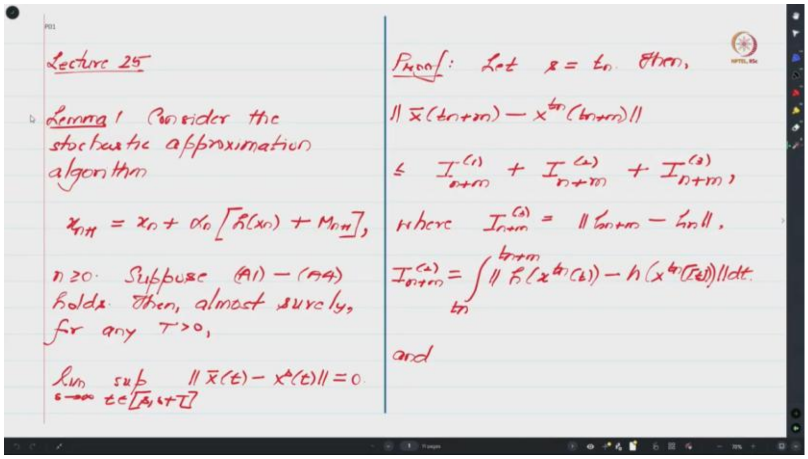
Week 6

Lecture 25

Proof of the Key Lemma - Part II

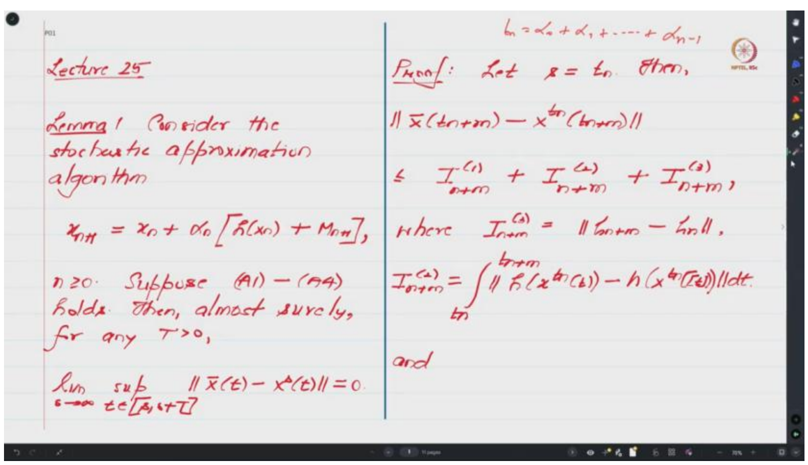
Hello and Namaste, everyone. Welcome to Lecture 25 of this NPTEL course on Stochastic Approximation. So, if you recall, in this and the previous week, we have been trying to show that the linear interpolation of the stochastic approximation iterates closely tracks a suitable solution of the limiting ODE. In this week, we have been focusing on proving a technical lemma that we used in the previous week to show that the limiting behavior of the stochastic approximation iterates is governed by suitable invariant sets of your limiting ODE.

So, with this background, let us begin today's discussion. So, recall that we have been trying to prove this lemma: Suppose we have a stochastic approximation algorithm given by the update rule x_n plus 1 equals x_n plus alpha_n times h of x_n plus m_n plus 1, recall that this refers to the noisy estimate of h of x_n. Right. And suppose that the assumptions A1 to A4 hold, then almost surely, for any capital T greater than 0, if you look at the linear interpolation of your X_n's, and the solution of your limiting ODE, which starts at x bar of s at time s, and if you look at the distance between these two trajectories for the time window between s and s plus t, and then take s to infinity.



Then our claim is that this quantity goes to 0 as n tends to infinity. And we have proved bits and pieces of this result in the previous two lectures. We will continue that discussion and try to finish off this proof in today's class. And let us quickly recall what we have done so far. So, you see that there is S tending to infinity.

To begin with, let us presume that S takes the special value of Tn. Recall that Tn equals alpha 0 plus alpha 1 plus dot dot dot plus alpha n minus 1. Is this okay? So, basically, this is your sum of these step sizes, right? And then what we had done last time was to compare the distance between x bar Tn plus M for some M greater than or equal to 0. Right.



And, you know, we wanted to compare this X bar Tn plus M with X Tn Tn plus M. So X Tn Tn plus M is basically the solution trajectory of your limiting ODE, which at time Tn. passes through X bar of Tn, which is basically your Xn. So, this is the limiting ODE which, at time Tn, passes through Xn, and we wanted to compare the distance between these two trajectories. In our previous class, we showed that the distance between them is upper bounded by three quantities. Where the third quantity is basically, you know, comes from your noise term or the cumulative noise part, and this is the norm of zeta n plus M minus zeta n, right?

So, here we work with the Euclidean norm, and since we are throughout working with the same norm, I am not emphasizing by putting 2 over here. The second term over here is basically the integral from Tn to Tn plus M of the distance between X Tn t minus X Tn square bracket t, the norm of this times dt. So, the way to interpret this is that this is the error that occurs due to discretization. Let me elaborate. The first thing to note here is that this corresponds to the solution trajectory that, at time Tn, passes through Xn.

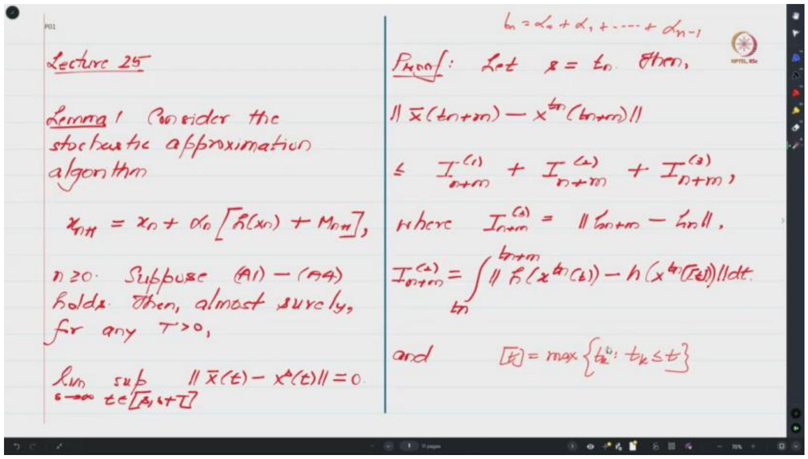
This also corresponds to the same solution trajectory but evaluated at some different point in time. In particular, your square bracket t is the max of tk such that Tk is less than or equal to T. Is this okay?

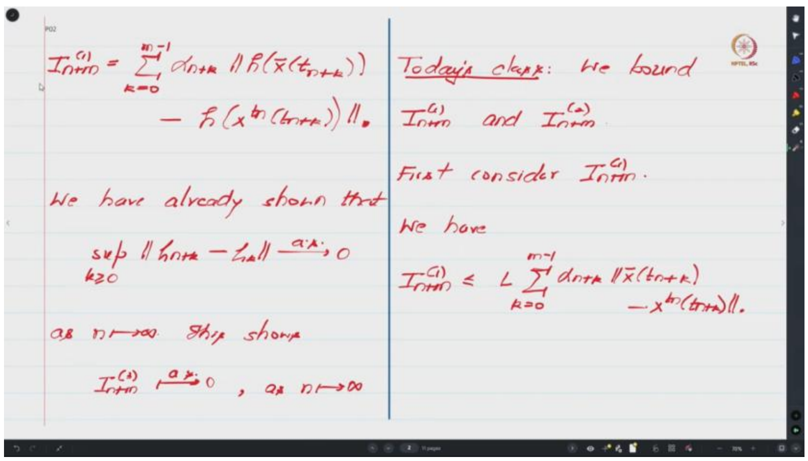
$$\left|\left|\bar{X}\,(t_n + r_n) - X^{t_n}(t_{n+m})\right|\right| \leq I^1_{n+m} + I^2_{n+m} + I^3_{n+m}$$

$$I^3_{n+m} = \left|\left|\zeta_{n+m} - \zeta_n\right|\right|$$

$$I^2_{n+m} = \int_{t_n}^{t_{n+m}} ||\zeta(x^{t_n}(t) - h(X^{t_n}[(t)])\,)||dt$$

So, it basically is the largest Tk-type time instance which is less than or equal to T. And your i n plus 1 m is the sum from k equals 0 to m minus 1 of alpha n plus k times the distance between h evaluated at x bar t n plus k and h evaluated at x t n t n plus k. So, if you look at this expression and if you look at the left-hand side over here, I hope you agree that they are of a similar nature in that they compare the value of x bar

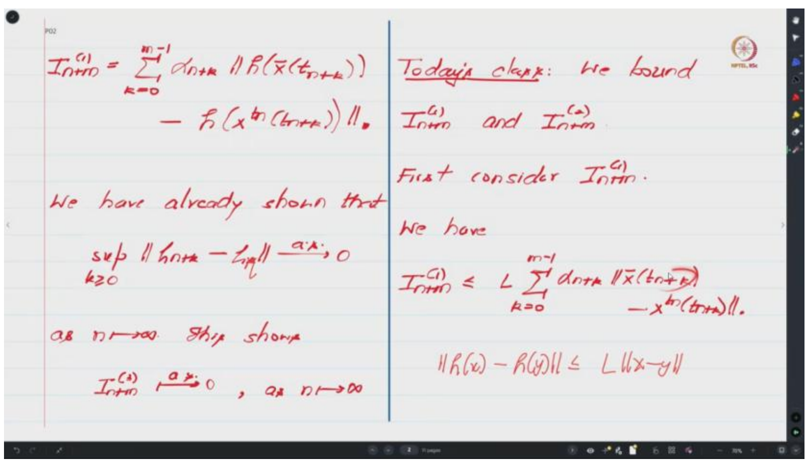


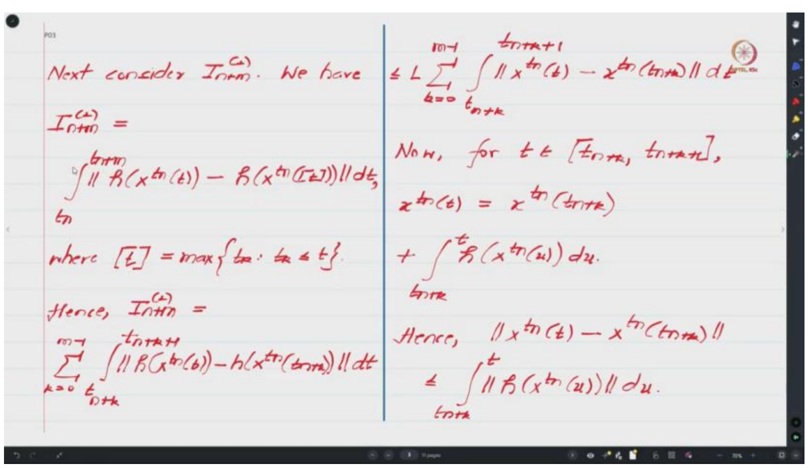


to the value of x tn at the same time instance. Of course, at i n plus 1 m, there is an additional h, but we will soon see how to get rid of this h and express this error directly in terms of the difference between x bar tn plus k and x tn tn plus k. And in our previous class, we have already shown that the third term, that is i n plus 1 raised to round bracket 3, which concerns your noise term. If you look at the supremum of k greater than 0 of zeta n plus k minus zeta n, this should be zeta n over here. So, this distance actually goes to 0 almost surely.

So, in some sense, it tells us that the third term is negligible asymptotically, and in today's class, we will therefore focus on the first two terms. To begin with, let us look at the first term, which is given over here. Now, what we will do is we will make use of the fact that your function L is Lipschitz continuous. To show that or to use the fact that h of x minus h of y is less than or equal to L times x minus y. So, we will use this fact, which comes from your A1 assumption, to conclude that this distance is upper-bounded by L times this distance. So, we now have an expression which mirrors the expression that we have over here, except for the fact that the arguments to x bar and x tn are different.

To begin with, we had tn plus m, whereas here we have tn plus k. However, notice that these terms are of a similar nature, and later on, we will see that whenever we have such expressions, we can invoke the Grunewald inequality to obtain some bounds. on the distance between X bar T n plus m and X T n T n plus m. So, for the time being, this is the intermediate bound on I n plus m m 1 that we are going to work with. So, now we move our attention to I n plus m 2. So, recall that the expression for I n plus m 2 is basically the integral from tn to tn plus m of the norm of the distance between x tn t and x tn square bracket t. So, observe that in I n plus m 1, we compare the linear interpolation of the stochastic approximation iterates with the solution trajectory of the limiting ODE. However, in I n plus m 2,



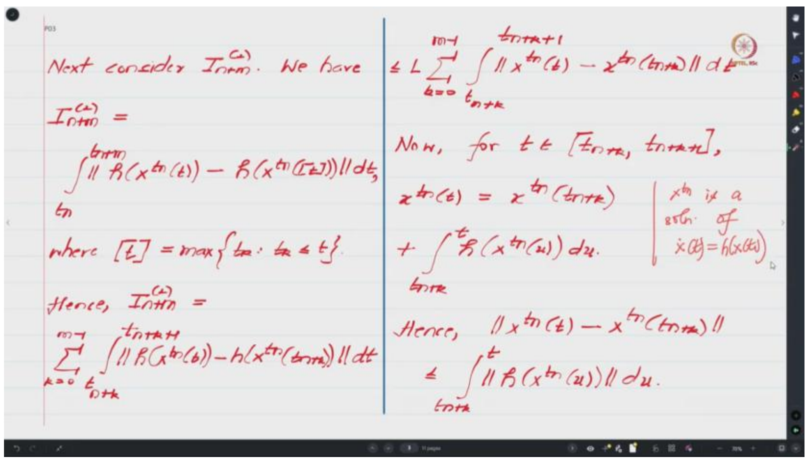


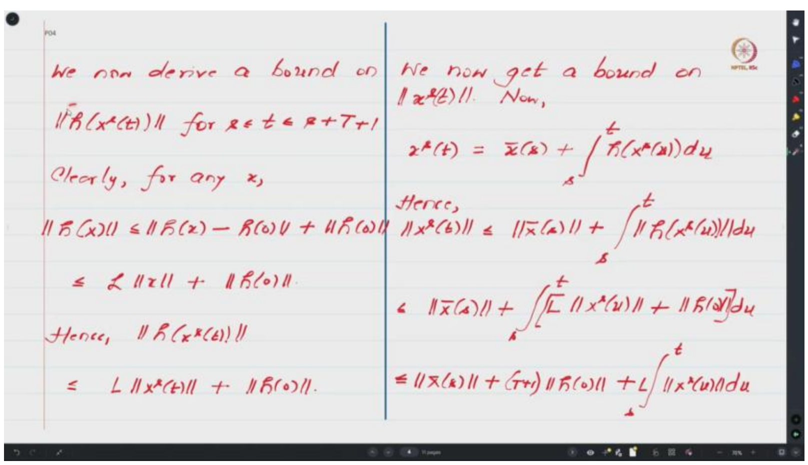
notice that both terms concern the solution trajectory of the limiting ODE itself, right? So now what we are going to do is we are first going to come up with a decomposition of this integral. Here, the integral starts from Tn and goes all the way up till Tn plus m. So what we are going to do is we are going to break this integral so that the integral now goes from Tn to Tn plus 1, then Tn plus 1 to Tn plus 2, and so on, and we will end up with something like this. So, notice that I have just broken down the integral, and when k equals 0, you have Tn to Tn plus 1. When k equals 1, you have Tn plus 1 to Tn plus 2, and so on, right? Now, when your little t belongs to Tn plus k to Tn plus k plus 1, right?

One can see that the square bracket T expression that we have over here translates to T n plus k, right? And hence, wherever I had square bracket T, I have replaced it with T n plus k for the case where T lies between T n plus k to T n plus k plus 1. So, that is why I end up with this expression. And again, I can use the Lipschitz continuity of H to conclude that this norm over here is upper bounded by L times the distance between the inputs to H. So, in this case, the input to H is X Tn of T and X Tn Tn plus K. So, we have something like this over here. And hence, this expression over here is upper bounded by what we have over here.

Is this okay? So, now... In the rest of the discussion, what we are going to show is that Because of your assumption A4, this expression is upper-bounded by some constant times alpha n plus k. And on top of that, the distance between t n plus k plus 1 and t n plus k is again alpha n plus k. So, together we will see that this term over here is upper-bounded by some constant times alpha n plus k squared. So, because we have this square of the step size over here, and if you recall in our assumption A2, we had presumed that your step sizes were square summable.

So, we will use that fact along with the fact that this expression is upper-bounded by alpha n plus k squared. We will use these two facts to show that this whole expression goes to 0. Now, let me formalize this. So, now what we are going to do is, for t between tn plus k to tn plus k plus 1, observe that because this x and t is a solution trajectory of your limiting ODE, we can write x tn of t equals the value of the solution trajectory at time tn plus k plus the integral from tn plus k to t of x tn u du. This just follows from the fact that your x tn is a solution of x dot t equals h of x of t. So, this just follows from this fact. And hence, if you take this expression to your left-hand side and take the norm, which is what we have done over here, we will see that this expression is upper-bounded by the integral from tn plus k to t times the norm of this expression over here. So, notice that this expression is precisely what you have over here. And in the rest of the discussion, what we are going to show is that because of your assumption A4, this expression is almost surely bounded by a constant, and hence this whole integral is upper-bounded by a constant times alpha n plus k. Let us see how we do that. So, as I said, we are going to derive a bound on this, and for the time being, we are going to presume that this T lies between S to S plus T plus 1.





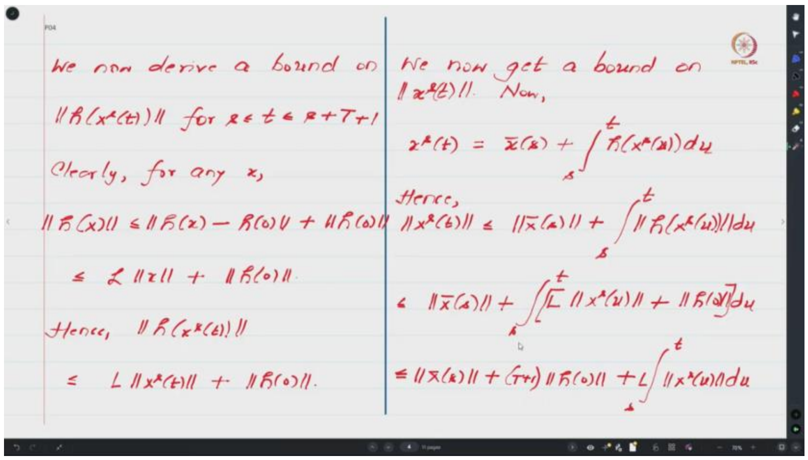
Whatever be your S we are going to look at a window of size T plus 1 within which we will try to get a bound of this. Now, notice that there is a plus 1 here bear with me for a few minutes I will soon explain why we require this plus 1 over here. At this point you know we are going to just try and bound this expression for t which lies within this window which is slightly larger than what we require. So, in our result we require that the window be of size capital T, but here we are working with the window of size capital T plus 1. And you will soon see why I am saying that.

Now, you know, for any x, I hope you agree that by a simple triangle inequality, the norm of h of x is upper bounded by the norm of h of x minus h of 0 plus the norm of h of 0. Right now what I have basically done is I have added and subtracted H of 0 and then invoke triangle inequality and hence we end up with the norm of this plus the norm of this expression. Now this expression is a constant with respect to X. So, we will not worry about this whereas by invoking the Lipschitz continuity of H one can conclude that this expression is upper bounded by L times norm of X. So, if we now substitute this special value of excess of T in place of X. One can now conclude that the norm of H excess of T is upper bounded by L times which is what we have over here times norm of excess of T plus norm of H of 0, right?

So, recall our goal, our goal is to show that these expressions whatever be your value of s and t are upper bounded by some constants right. So, towards showing that what we are going to do is we are now going to try and obtain a bound on excess of t norm of excess of t right now let us recall what is excess of t it is the solution to your limiting ode that passes through x bar of s at time s so by using that fact one can see that excess of t equals x bar of s plus the integral from s to t h excess of u du. This just follows from the fact that your excess of t is actually a solution of your limiting OD which at time s passes through x bar of s. So, now if I invoke

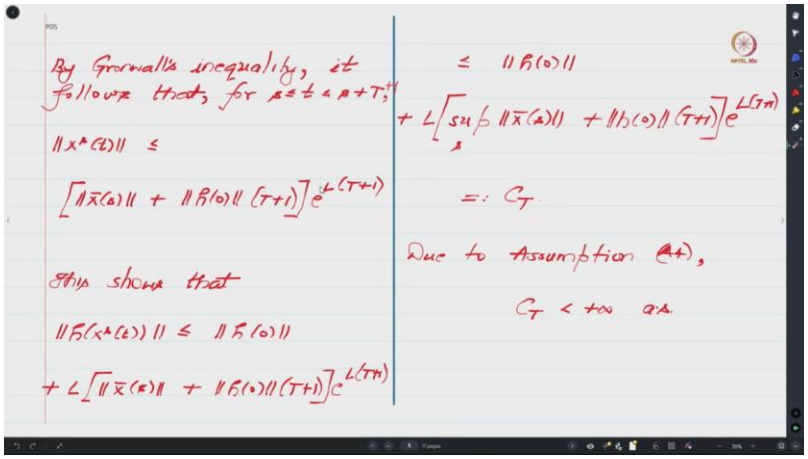
You know the triangle inequality—one can see that the norm of this expression, that is, x of t, which is what we have over here, is upper bounded by x bar of s plus the integral from s to t of the norm of x. S of u du, right? This is just a simple triangle inequality, right? And now we have already shown here that, you know, for any x, the norm of H of x is upper bounded by this expression. So, I am going to replace this expression with L times the norm of XSU plus the norm of H of 0, right?

Times du, right? And then, by using the fact that this is a constant, you know, I can pull it out and multiply it by the distance between t minus s. But notice that t is chosen so that the distance between t minus s can at most be t plus 1. Hence, h of 0's integral from s to t is upper bounded by t plus 1 times the norm of h of 0, right? And whatever is the second expression over here, I bring it down over here as it is. So, let us just summarize what we have done.



Our goal was to show that there is, you know, this quantity is uniformly bounded for S and T. And in order to do that, we first made use of the Lipschitz continuity of H to show that this norm holds. And then what we have shown is that because you have the norm of x of t, we have now shown that x of t is upper bounded by some expression like this plus L times the integral of some expression. Now notice that the expression on the left here and the expression that we have over here, they are of the same nature. Whenever we have an implicit relation of this form, in your mind, the Grönwall inequality should start ringing. This is the kind of relation where Grönwall inequality can be used to get an explicit upper bound on the expression on the left-hand side.

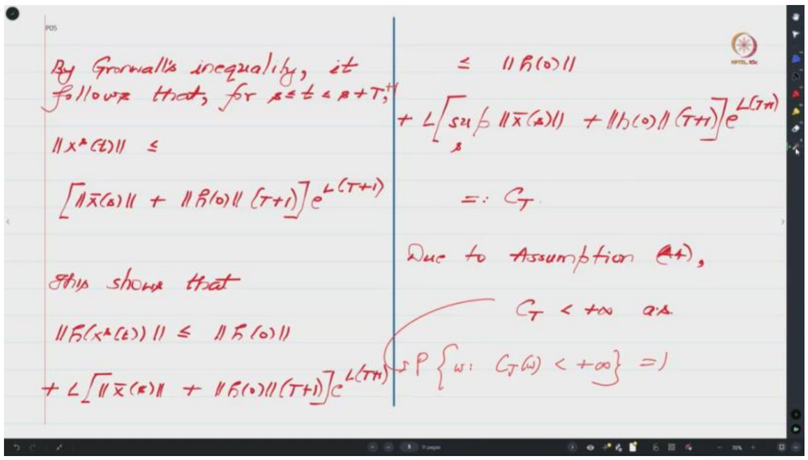
So, we will soon see how we can do that. So, we use Grönwall's inequality and show that your x, x of t's norm is actually upper bounded by this constant, which does not depend on t, times e raised to l times. So, the l actually comes from here, right? Times the distance between t and s. But the distance between t and s is again upper bounded by t plus 1. I should emphasize that over here. Right, and hence the whole expression is upper bounded by the constant plus e raised to l times



capital T plus 1. So, notice that the right-hand side does not depend on T anymore. So, that is what we have done here. Also, observe that previously we had an implicit inequality—that is, we had excess of T, the norm of this is upper bounded by some expression involving the norm of excess of U. So, in this sense, this norm was upper bounded by some expression involving the previous value of this trajectory itself. So, in that sense, this inequality was implicit in nature. But now, by invoking Grönwall's inequality, we have made this inequality explicit—that is, on the right-hand side, we don't have any expression that looks like this form, right? So, now recall that your norm of H excess of T was upper bounded by this, and we have now obtained a bound on this.

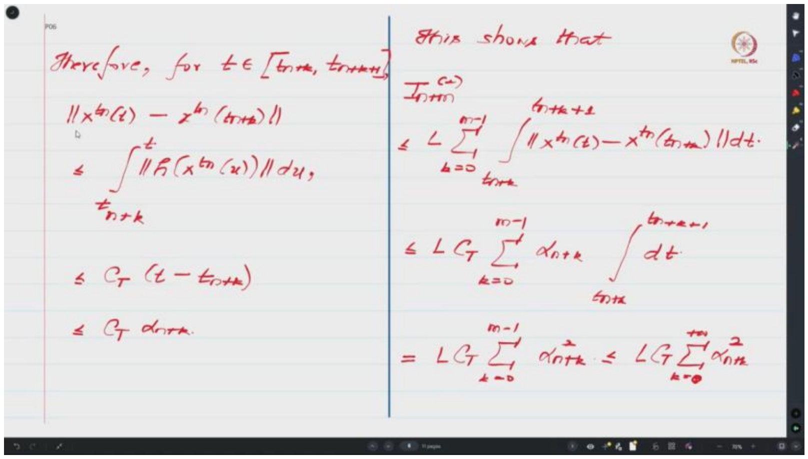
So, hence, substituting the bound on this in here, we can conclude that the norm of H excess of T is upper bounded by norm H of 0 plus L times—wherever your norm excess of T was there—we replace it with this expression, and hence we conclude this thing, right. So, now what we do is, wherever you have norm X bar of S, notice that this expression corresponds to your linearly interpolated stochastic approximation trajectory's value at time S. So, we replace this expression with its supremum over all possible values of S. Now, we know that this expression, because of assumption A4, is almost surely bounded. Hence, this whole expression that we have on the right-hand side is almost surely bounded. So, let us denote this expression by some random variable Ct, so that we can formally conclude that, due to assumption A4, your Ct random variable is less than infinity almost surely.

So, again, this means that your Ct is actually a random variable, and the collection of omega and the collection of omega such that your CT omega is less than infinity has probability equal to 1. So, this follows because of our assumption that, in A4, your stochastic approximation iterates are almost surely bounded, which implies that this expression over here is also almost surely bounded. Okay, so let us summarize what we have done so far. We did a lot of algebra, but at the end of the day, what we have shown is that there exists some constant, right?



Which is almost—I should not say constant. This is like, on every sample path, this is a constant, right? But on different sample paths, the value of this constant could be different, right? And what we have shown is that, for every sample path, this expression, which depends on s and t, right, is upper bounded by a quantity that does not depend on little s and little t. So, this quantity uniformly upper bounds this expression over here for different values of s and t, right? So, this constant we can invoke now to obtain a bound on

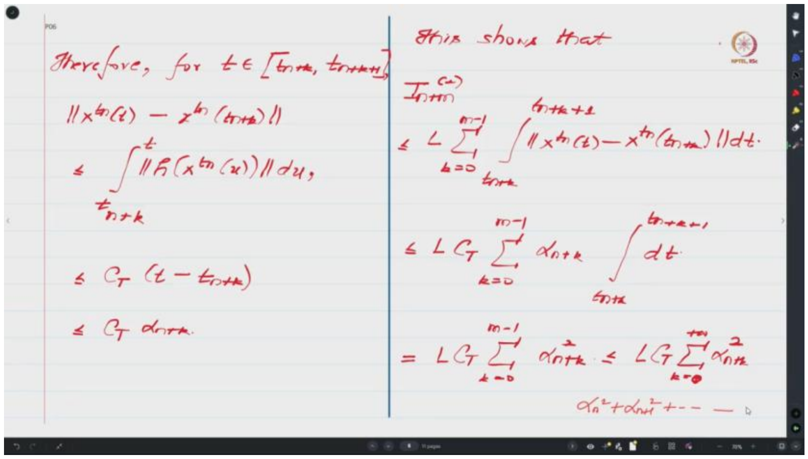
the norm between x t and t and x t and t n plus k. So, recall that the distance between them we had earlier shown is upper bounded by some expression like this, right, and our goal was to show that this expression is upper bounded by some constant times alpha n plus k. So, let us see how we can do that. So, in our previous slide, we have shown that the norm of this expression is upper bounded by some sample path-dependent constant, but independent of these values of t, n, and u. So, we can come up with Ct over here, right. And once we replace this with Ct, this whole integral is upper bounded then by Ct times T minus T n plus k, right? And since T lies between T n plus k and T n plus k plus 1, one can conclude that this expression is upper bounded by alpha n plus k, right?



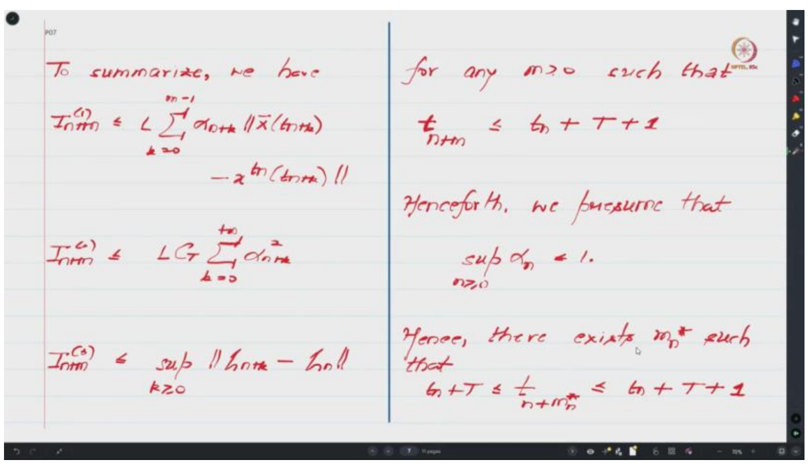
So, let us see how this is helpful. So, recall that the bound on i n plus m2 that we had derived was that this expression is upper bounded by L times the sum of this integral from tn plus k to tn plus k plus 1 of this distance. So, this is the distance which we have upper bounded over here. And now what we have shown is that this distance is upper bounded by CT times alpha n plus k. And then we have an integral from Tn plus k to Tn plus k plus 1. Hence, this expression over here is again upper bounded by alpha n plus k, finally giving us this alpha n plus k squared.

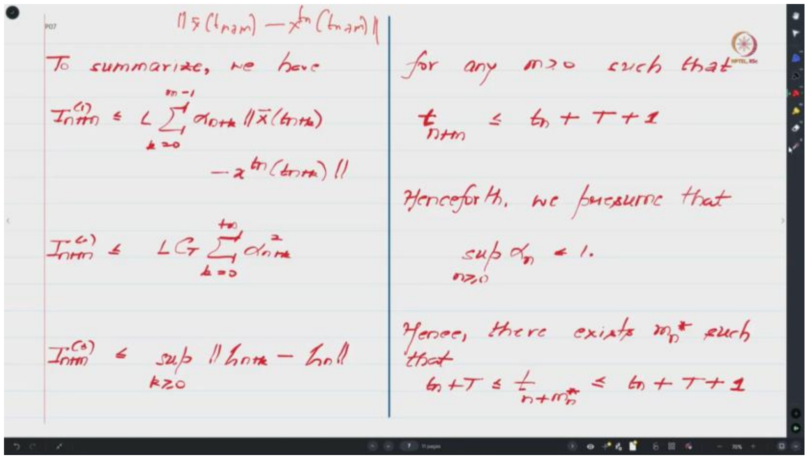
So, in this case, we sort of write a sum starting from k equals 0 to m minus 1, but since this sum is made up of non-negative terms, one can upper bound this expression by the sum where the upper index is actually replaced by plus infinity. So, let me summarize what we have done so far. We have said that on every sample point, that is on every omega, the value of I n plus m2 is upper bounded by some sample path-dependent constant times the sum of these squares of these step sizes. Now, if you substitute k equals 0 here, this is alpha n squared and so on. So, which means that this sum

is actually of the form alpha n squared, alpha n plus 1 squared, and so on. And we know that the sum of alpha n squared starting from n equals 1 to infinity is actually bounded. Hence, this is like the tail of an absolutely summable series. And since this is the tail, as your n becomes larger and larger, the value of this tail is going to go down to 0. So, in this sense, you can see that this expression goes to 0 as n tends to infinity, which is what we wanted to show.



So, let us summarize what we have done so far. So, to summarize, we have shown that the bound on I n plus 1 m is actually upper bounded by L times the sum of terms which look very similar to, you know, what is there on the left-hand side. So, recall the term on the left-hand side was actually x bar tn plus m minus x tn tn plus m. So, this is the expression that we started out with, and we had, you know, shown in the previous class that this is upper bounded by this quantity plus this quantity plus this quantity. In today's class, what we have shown is that this quantity is upper bounded by some constant times some sum.



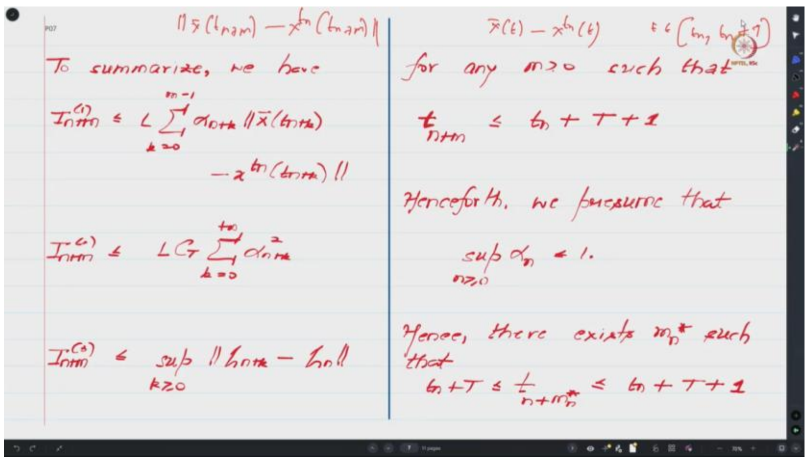


involving terms which look very similar to this, except that here the arguments are tn plus m, whereas the arguments here are tn plus k, right, for different values of k. However, this expression looks very similar to this. Again, we will show you later how we can use a version of Grönwall's inequality to translate this implicit inequality to an explicit inequality, but let us hold that thought for the time being. Now, the second term that upper bounded this expression was IN plus M2, and we just now showed that this expression is upper bounded by a a sample point-based, you know, constant, right, times the tail of the, you know, sum of the squares of your step sizes. So, I am saying it is the tail because you have n plus k over here. So, when you substitute k equals 0, you start from alpha n square and so on.

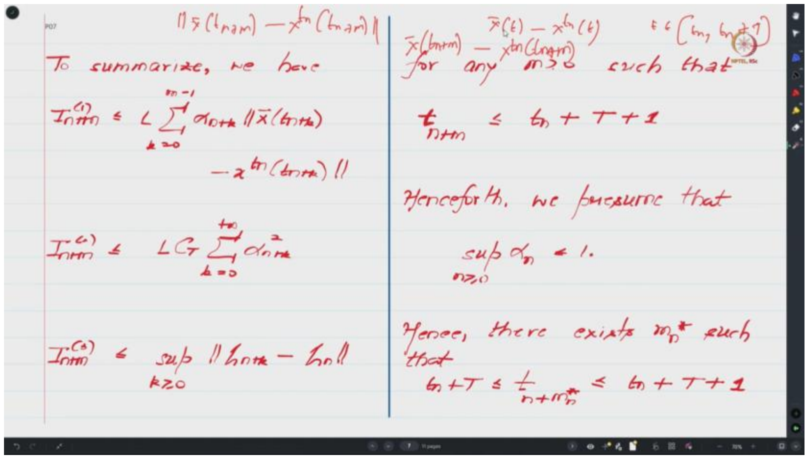
So, since this sum starts from alpha n square, this is the tail of a summable series, and hence, as n tends to infinity, this expression will go to 0. And in the previous class, we had shown that i n plus m 3 is upper bounded by sup of k greater than or equal to 0, the norm of zeta n plus k minus zeta n, and because your martingale converges, right, again, this is looking at the, you know, the tail portions of a convergent sequence And invoking the fact that, you know, every convergent sequence is also Cauchy, one can show that, you know, this expression actually goes to 0 as n tends to infinity, right? Okay, now, you know, we have shown this, you know, for any m, in particular, we have shown this for m, which satisfies the fact that Tn plus m is less than or equal to Tn plus T plus 1, right? Is this okay?

So, now I will try to justify why we need this plus 1, right? So, why we need this plus 1 is that eventually, you know, our discussion has to involve what happens between X bar T and X T and T, right? Where T can be some arbitrary value. Between T can be some arbitrary value between Tn and Tn plus capital T. So, this is what we require in our result. This is what we need to do in our result.

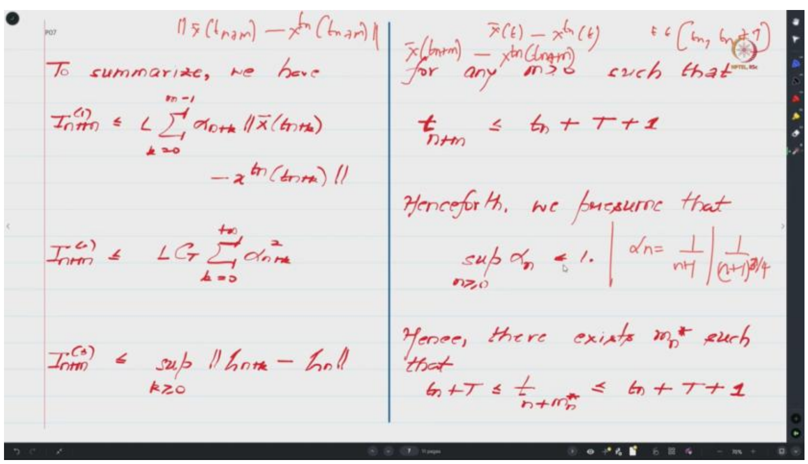
And in order to do that, what we are going to do is we are going to make use of the distance between X bar Tn plus M and X Tn Tn plus M. Right. And what we are going to do is we are going to identify some values of m so that t is close to these tn plus m. So I will show you how to do that. Right. And sometimes what will happen is that the closest tn plus m may be above t as well.



And since it is above t, it is possible that for some little t which is close to tn plus t, to find a tn plus m type expression close to such a t, we may want to go slightly beyond this tn plus t as well. And hence, we are working with a window which is slightly wider. And what we were going to do is, henceforth we are going to presume that your step sizes are upper bounded by 1. For example, a step size of the form alpha n equals 1 over n plus 1 or 1 over n plus 1 raised to 3 over 4 and so on. So, these kinds of step sizes one can see that they are all, you know, always upper bounded by 1.

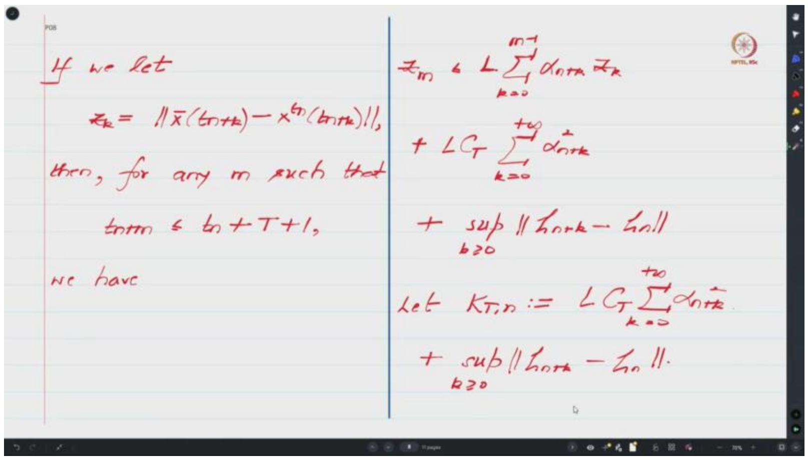


And even if it is not upper-bounded by 1, if it is upper-bounded by some constant, then we just have to replace that constant over here. So, in that sense, this is not such a big assumption. So, let us presume that the supremum of the step sizes is less than 1, which implies that there exists an M_n^* such that T_n + M_n^* is between T_n + T and T_n + T + 1. So, as I said, for little t's which are very close to T_n + T, we want to come up with a time index of the form T_n + k, right?



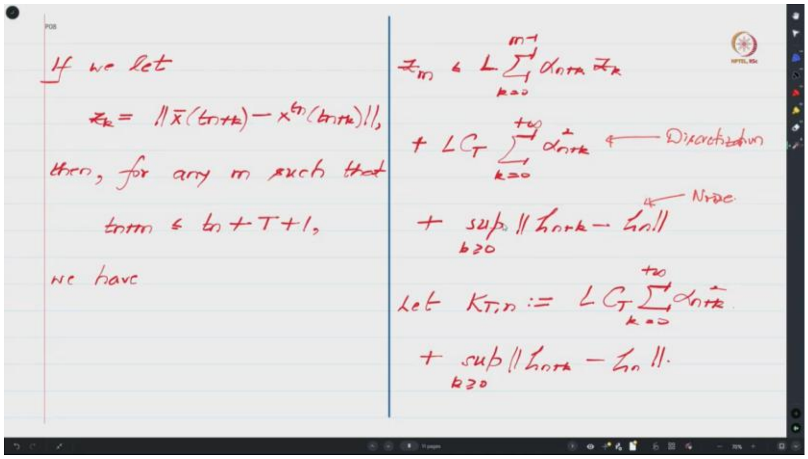
And this supremum of these step sizes being upper-bounded by 1 ensures that one can find M_n^*, which depends on T (capital T), such that T_n + T and T_n + T + 1 upper-bound this, right. This is easy to see from the fact that the distance between T_n + T and T_n + T + 1 is 1, right, and your step sizes are upper-bounded by 1, and hence, if you keep looking at indices of this form, they can at most increase by one. So, if you have two time instances which are one distance apart, there has to be some T_n + k sitting between them, and the k that sits between them is what we refer to as M_n^* over here.

So now, let us condense what we have shown so far. If we let Z_k be the distance between your linear interpolation of stochastic approximation trajectories and the limiting ODE solution at time T_n + k, then what we have shown so far can be succinctly written as Z_m, which is what we have on the left-hand side over here. So, Z_m is upper-bounded by the bounds on I_{n+1}^(1), I_{n+1}^(2), and I_{n+M}^(3), right. So, the bound on I_{n+M}^(1)

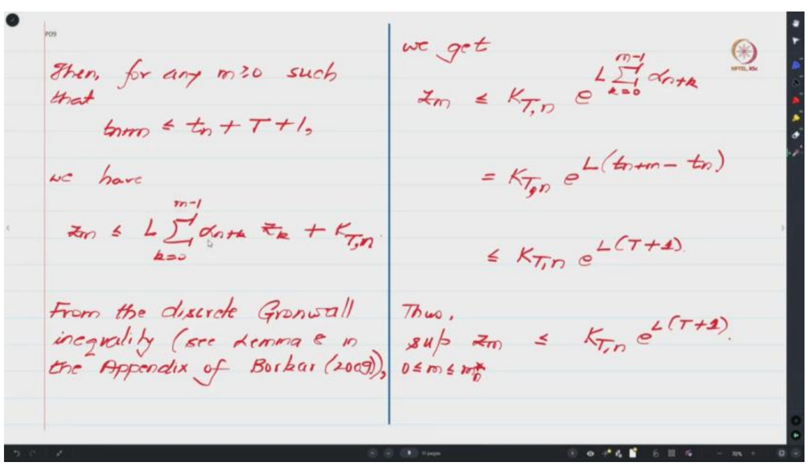


Again, when we replace these expressions over here by Zk, we end up with L times the sum from k equals 0 to m minus 1 of alpha n plus k Zk. Now, the second term we had shown is upper bounded by some sample path-dependent constant times the tail of the squares of these step sizes, and the third term is upper bounded by the tail of your convergent martingale sequence. So, again, let us connect to what we were discussing before. This is the noise that arises due to discretization or the error that arises due to discretization, and this is the error that arises due to your noise.

So, we have upper bounded the discretization error by the squares of the step sizes. And the noise term we have upper bounded by this expression. So, what we are going to do is we are going to, you know, denote this whole thing as kTn, right? So, this whole thing as kTn. So, observe that this index, when you substitute k equals 0—I mean, the index of this term when you substitute k equals 0—is n; hence, this expression actually depends on n.



And, of course, it depends on T, whereas this expression, again because of the supremum, depends on N, and we are highlighting the fact that these two terms depend on T and N by putting these variables as you know, subscripts of this constant K. So, KTN is basically these two things, like the sum of these two expressions, and we are putting these T and N in the subscript to denote that these expressions actually depend on capital T and N. Again, I would like to highlight that this CT is a sample path-dependent constant, which means that on different sample points, the constant could take different values. However, for that sample point, it will be a constant with respect to the different values of T and S and so on and so forth. So, now, you know, once we have introduced this KTN expression, one can show that we have ZM is less than the sum of L times the sum from K equals 0 to M minus 1 of alpha N plus K times ZK plus KTN, right?



So, again, see that you have an expression on the left, which is upper bounded. By an expression which involves Z case. So, we have an implicit inequality over here. So, we can now make use of the Grönwall inequality. The Grönwall inequality that we had discussed in class was for continuous time instances.

Here, however, we will need to invoke a discrete version of the Grönwall inequality. So, the proof of the discrete Grönwall inequality can be done very similarly to the continuous Grönwall inequality that we had proved in class, and I refer the audience members to look at Lemma 8 in the appendix of Borkar's textbook to find a proof of this discrete Grönwall inequality. But by invoking the discrete Grönwall inequality, one can show that this expression is upper bounded by KTN times E raised to L. So, this L actually sits over here, and whatever we have over here, I am going to write it this way, right? So, you have this sum of alpha N plus K, right?

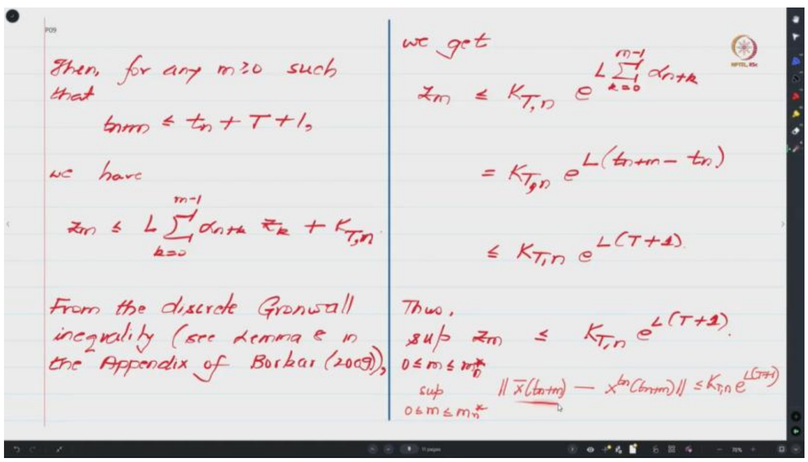
And by the definition of your Tn and Tn plus m, one can see that this expression is actually equal to Tn plus m minus Tn. Hence, we can conclude that Zm is upper bounded by this expression. So, the nice thing about this expression is that notice that Zm is on the left-hand side. On the right-hand side, you do not have any expression that looks like Zm. So, this, in that sense, is an explicit inequality.

Whereas, what we had over here was an implicit inequality relation that is, in some sense, the benefit of working with Grönwall's inequality. So, now we know that the M's we have chosen are such that Tn plus M is upper bounded by Tn plus T, and we had taken some slightly larger window. Hence, the gap between Tn plus M and Tn can be upper bounded by T plus 1. Hence, one can finally conclude that ZM is upper bounded by KTN times E raised to L times T plus 1. So, let us pause here for a second and ponder what we have derived.

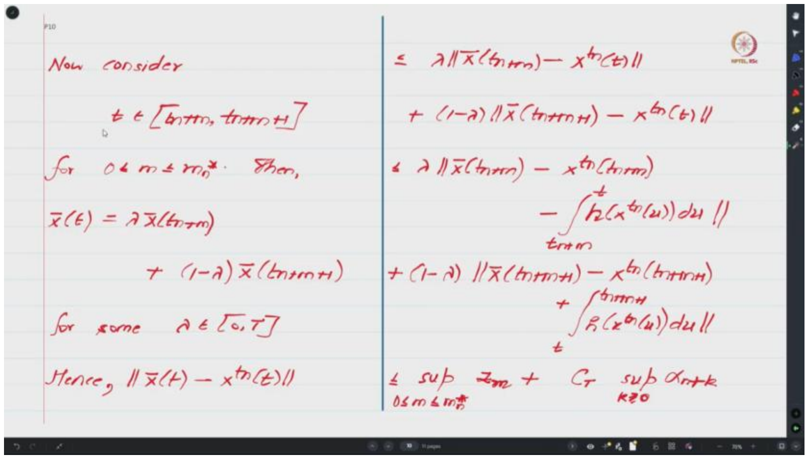
So, if you fix T, that is the length of your window, then this expression will be a constant. This expression is also a constant with respect to m, but as n tends to infinity, this expression goes to 0. So, that is the nice thing we have shown over here, and we can finally conclude that because this expression over here does not depend on m. Hence, we can take the supremum of ZM, where M goes from 0 to MN star, right? Because we need to ensure that this TN plus M is upper bounded by TN plus T plus 1.

Hence, we need this MN star over here. So, we can conclude that the supremum of the ZMs is upper bounded by this expression over here. Is this okay? So, let us recall what this Zm is. So, what we have managed to show is that the supremum of the norm of X bar Tn plus M minus X Tn Tn plus M, where 0 is less than M less than Mn star.

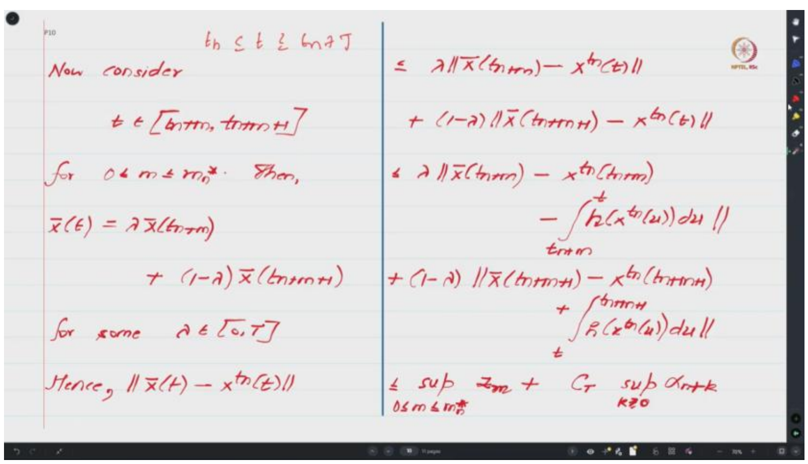
This expression is upper bounded by K T n times e raised to L T plus 1. This is what we have managed to show, right? So, in other words, we have managed to show that the distance between the value of your stochastic approximation trajectory at time instance T n plus m and the stochastic value of your limiting ODE solution trajectory at time T n plus m for any m that lies between 0 and m n star is actually upper bounded by K T n times e raised to L T plus 1, right?



So, this is what we have managed to show. However, in the result, what we need to show is we need to compare the distance between x bar t and x t at t, okay? Now, that is again a bit of algebra, but that is not very hard, and I will quickly show how to do that, okay? So, let us consider some arbitrary value of t, right? And as I told you, whenever we have an arbitrary value of t, we have to, in some sense, identify some time instances of this T n form, which upper and lower bound it, okay?



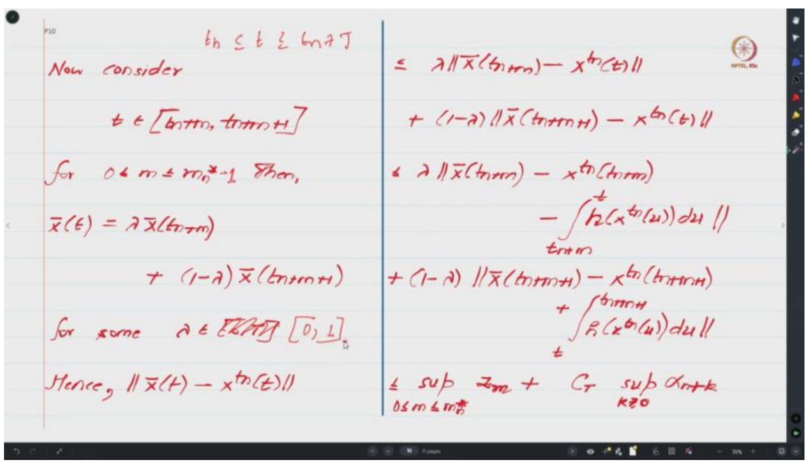
So, without loss of generality, let us say we, so we want to, you know, work with this So, T n less than T less than T n plus capital T. So, this is the T that we want to work in the result, right? Now, for any T, my claim is that you can find an m which is between 0 and m n star, right? Such that T lies between, in fact, I should say less than or equal to m n star minus 1, okay? Such that, so let me just make sure m n star's definition is correct.



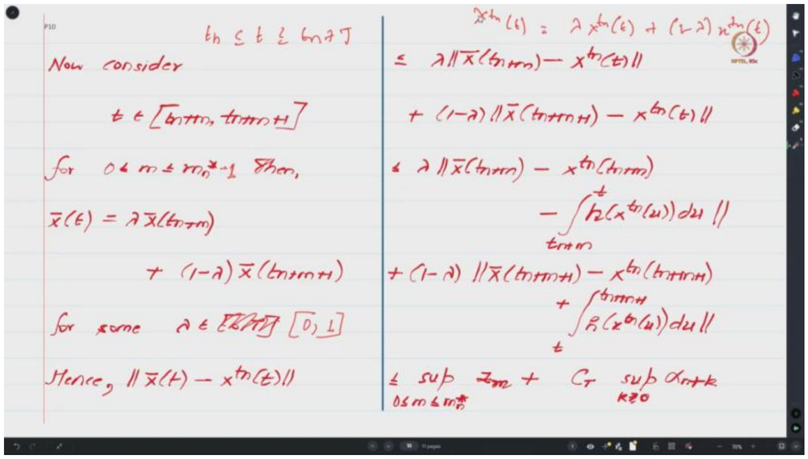
So, Mn star, yeah, Mn star actually goes beyond Tn plus T. Very good. Okay. So, my claim is that there exists M. between 0 and Mn star minus 1 so that if T lies within this window, right, then T will lie between Tn plus M and Tn plus M plus 1. Okay, so this is something very straightforward and you can check it on your own.

So, now for this arbitrary value of t, the goal now is to compare the value of X bar of t with X tn of t. So, so far we have compared the value of X bar and X tn at these special time instances. Now, we are trying to generalize it to any t between tn and tn plus t. So, for that we first identify this window t n plus m and t n plus m plus 1. So, for such a t because we are working with linear interpolation one can show that x bar of t can be expressed as a convex combination of the values of x bar at t n plus m and t n plus m plus 1. So, that convex combination we are expressing in the following way that is x bar of t is lambda times x bar of t n plus m plus 1 minus lambda times x bar of t n plus m plus 1 for some lambda between 0 and 1.

So, I should emphasize that this is not 0 to t rather this is 0 to 1. So, fact that lambda is between 0 to 1 makes this a convex combination and as I said since you are working with linear interpolations and t lies between this such a relation is obvious. And now let us compare the distance between X bar t and X t and t. right. So, now x bar of t has this expression.



So, I am just writing this as x bar t n plus m times lambda 1 minus lambda times x bar t n plus m, right. And since lambda plus 1 minus lambda is 1, I can express your x t n t as lambda times x t n t plus 1 minus lambda times xtn t. So, notice that lambda plus 1 minus lambda is 1 and these two expressions are one and the same. Hence, they should add up to give you this xtn t. So, wherever there was xtn t, I wrote it in this form and this x bar t, I wrote it in this form by invoking the fact that Your x bar is obtained by linear interpolation and hence a simple triangle inequality shows that this distance is upper bounded by lambda times x bar t n plus m minus x t n t plus 1 minus lambda times x bar t n plus m plus 1 minus x t n t.

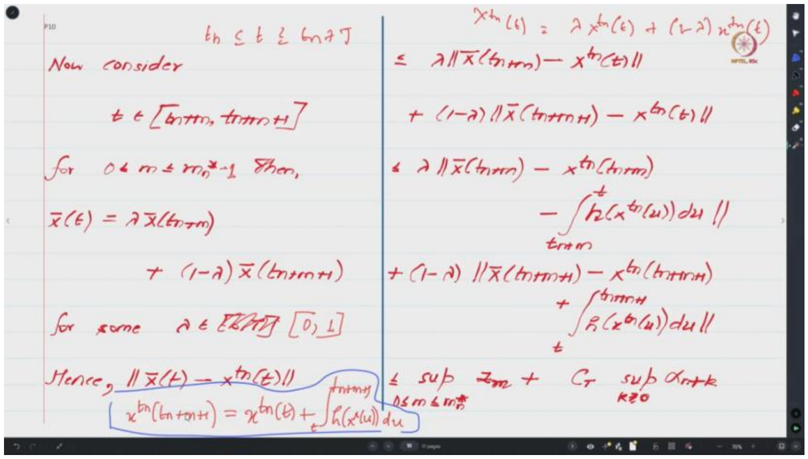


So, now one can ask: can we directly upper-bound these expressions? Notice that in our calculation so far, we do not have a bound on this expression. Specifically, we have a bound on this expression only when T is of the form T n plus m. And similarly, here T is of the form T n plus m plus 1. However, this is some generic value of T which lies between these. So, the next question that we have to handle is how do we get T n plus m here and T n plus m plus 1 over here.

But that is straightforward because this is the solution trajectory of your limiting ODE. We know that Xt and t is the sum of the value of your solution trajectory at tn plus m plus the integral from tn plus m to t H of tn u du, right? So, this just follows from the fact that this is a solution trajectory of your limiting ODE, right? So, I can write it this way, and similarly this expression over here, right? I can say that the value at xtn, tn plus m plus 1 is the value at time t plus this integral, right?

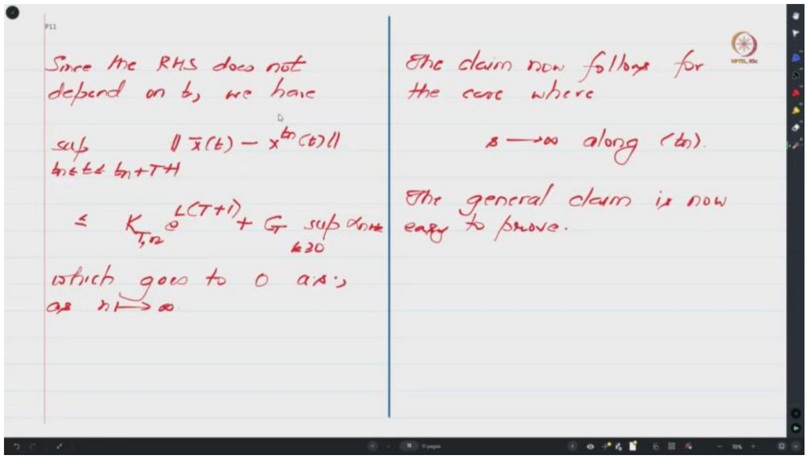
And since, you know, I mean, so let me see if I can write somewhere here. So, what I am saying is x. T n T n plus m plus 1 is equal to x T n T plus integral T to T n plus m plus 1 h x s of u d u. So, we can write such a relation like this. Okay. And we have X, T and T over here.

Right. So, wherever we have X, T and T, I am going to take this expression and subtract it from this expression and substitute it back over here. And that is exactly what I have done. Is this okay? So, now notice that this expression is something that we can bound.

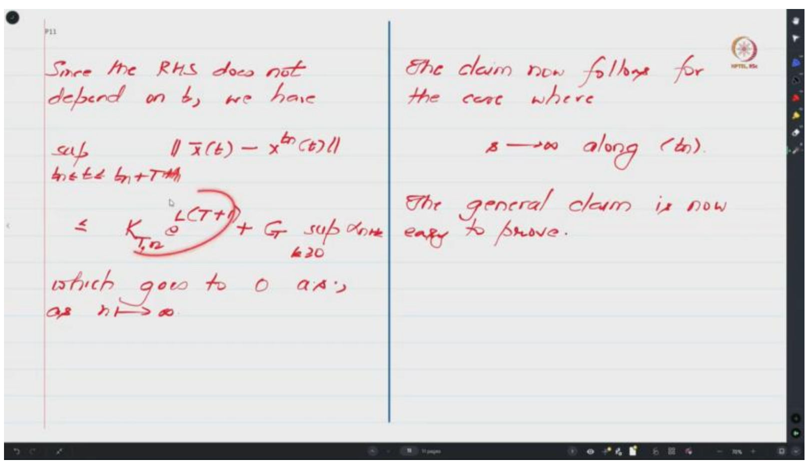


Similarly, this expression is something that we can bound. Right. And we have separately shown that this expression is upper bounded by C, T. times the distance between t to t n plus m and since the distance between t to t n plus m is upper bounded by alpha n plus m right and similarly this expression is also upper bounded by alpha n plus m and this is upper bounded by c t right and lambda and 1 minus lambda add up to 1. One can show that the norm of this expression and the norm of this expression is actually upper bounded by CT times alpha n plus m and alpha n plus m is upper bounded by the supremum of alpha n plus k from k going to 0.

So, in this sense, this expression now no longer depends on M, and whatever we had before, right? These two expressions—these are your ZMs, you know, ZM and ZM plus 1—and these expressions will be upper bounded by your supremum of ZM where M lies between 0 to Mn star. So, in this sense, both these expressions now no longer depend on your M. Right? And hence, they upper bound the distance between x bar t and x t and t for any t that lies between t n and t n plus t, okay? So, that is what I have written over here: that the norm of x bar t minus x t and t for t lying between t n and t n plus t. So, here I should make it

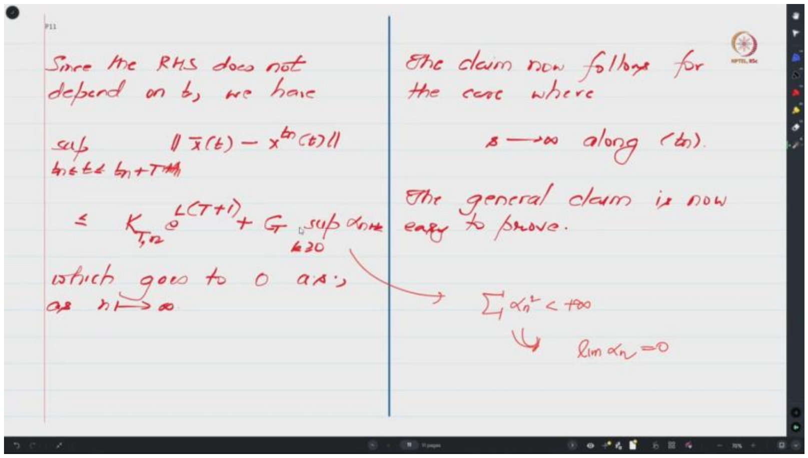


Tn over here, Tn plus T over here, because this is what was needed in the result, right? So, this upper bound—sorry, this expression—is upper bounded now by an expression like this, right? And this comes from the upper bound on the supremum of the Zm's, and this comes because we had such a term over here. Now, we know that both these terms are going to 0 as n tends to infinity.



The first term goes to 0 because ktn goes to 0, and the second term goes to 0 because your step sizes go to 0. So, why does this happen? Again, this happens due to the fact that the summation of alpha n squared is less than infinity. So, this forces that your limit alpha n is 0.

This fact itself forces that your limit alpha n is 0, and hence, if you look at the supremum of the tail, this expression will also go to 0 as n tends to infinity. So, from this, one can conclude that the supremum of the distance between X bar t and X t, and t within a window of size t, goes to 0 as n tends to infinity. So, now one can ask: have we finished proving the lemma? Well, we have finished proving the lemma for the case where s goes to infinity along this time sequence tn. So, notice that here there is tn.



However, to go from tn to general s is now very straightforward because recall that in all our discussions, we actually worked with a window of length t plus 1 whenever we were deriving the intermediate results, and, you know, working with such a result, one can, you know, easily show that whatever we have shown along tn, the same arguments can be generalized to prove for general s as well. So, with this, we come to the end of today's class. Let me quickly summarize what we have done in this class. So, in this class, we have finished the proof of Lemma 1, right?

And hence, also the proof of your main theorem, which talked about the asymptotic behavior of your stochastic approximation iterates. So, in Lemma 1, we needed to show that if you look at any finite window of length t, then the distance between your stochastic approximation iterates and your limiting ODE solution—the distance between them within this window—goes to 0 as the starting point of the window is taken to infinity. So, we have finished this proof. In the next class, what we are going to do is we are going to look at applications where these four assumptions can be shown to hold, and hence, we can, you know, discuss the convergence of those stochastic approximation algorithms.

So, until then, thank you. Goodbye.