STOCHASTIC APPROXIMATION: THEORY AND APPLICATIONS

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Week 4

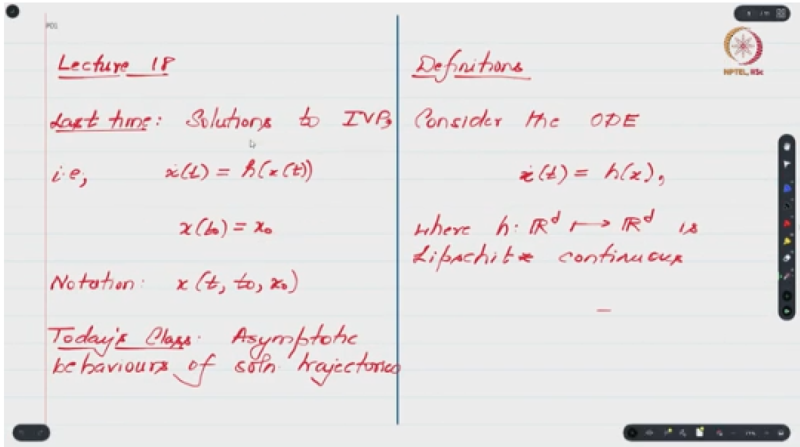
Lecture 18

Asymptotic Behaviour of Solutions to ODEs

Hello and Namaste, everyone. Welcome to Lecture 18 of this NPTEL course on Stochastic Approximation. Let us do a quick review of what we have been doing this week. So, the plan for this week is to go over some fundamentals of ordinary differential equations. You will soon see that we will be using suitable ordinary differential equations to predict and analyze the behavior of stochastic approximation algorithms.

So, that is what we will discuss in the forthcoming week. So, this week, our task is to get some basics of ordinary differential equations in order, and in the previous two lectures, we looked at something called initial value problems. Then we discussed the existence and uniqueness of solutions to this initial value problem. And then we also looked at or discussed the continuity of the solutions to this IVP with respect to the initial condition. So, this is what we have covered so far, and today we will study a bit more about the solutions to these initial value problems, okay?

So, recall that an initial value problem is an equation of this form, and this is the initialization condition, and we use the notation x of (t). x of (t, t0, x0) to denote the solution trajectory that satisfies this IVP. In today's class, we will try to understand the asymptotic behavior of such solution trajectories. In order to do that, we would need to introduce a few definitions. So, towards that, let us consider an ordinary differential equation of the form x dot of t equal to h of x of t. So, notice that here, I often would not write x of (t). So, this t over here is hidden, and it is done for simplicity of notation. But whether I write x of (t) or not, it is to be interpreted that there is an x of (t) over here.



So, let us consider an ODE of the form x dot of t equals h of x, where h is a mapping from Rd to Rd, and we will presume that this function h is Lipschitz continuous. Now, I hope you are able to understand and appreciate. The importance of this function h being Lipschitz continuous.

$$\dot{x}(t) = h(x(t))$$

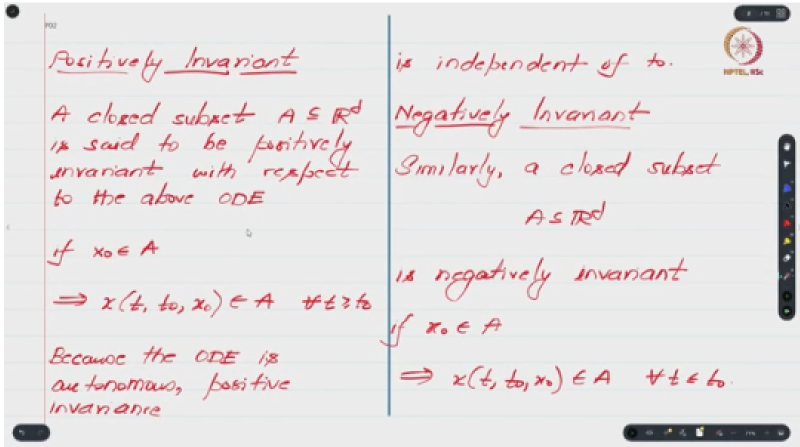
$$x\left(t, t_0, x_0\right)$$

$$\dot{x}(t) = h(x(t))$$

$$h: R^d \mapsto R^d$$

On one hand, this ensures that this initial value problem has a unique solution. On the other hand, it also ensures that the solution trajectories are continuous with respect to this initialization condition x naught.

So, as I told you, we require a few definitions, and one of the definitions that we will begin with is that of positive invariance. So, we will say a closed subset A of Rd is positively invariant with respect to this ODE x dot of t equals h of x of t. For every initialization x0 in A, that is, if you start the solution trajectory of this ODE in A, then that solution trajectory lies in A for all t greater than or equal to t0. So, notice that there is a greater than or equal to sign. So, this greater than or equal to sign and this positive, you know, are connected.



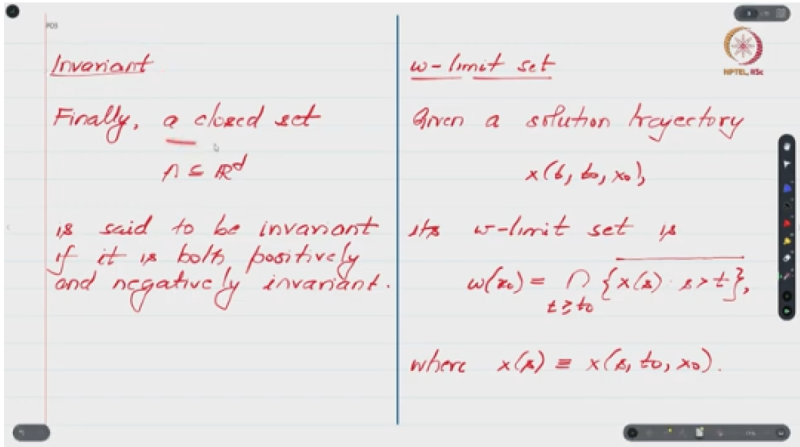
So, this positive is there because of this greater than or equal to sign. So, what does positive invariance mean? It means that, you know, if you start a solution trajectory of this ODE in this set, then for all t greater than or equal to t0, the solution trajectory will lie in A, right? And I would also like to emphasize that, you know, throughout this course on stochastic approximation, we will more or less be focusing on autonomous ODEs. And because we are focusing on what are called autonomous ODEs, the positive invariance definition is independent of this initialization t naught.

So, this T naught as such plays no role. Now, in this same spirit, we can define this concept called negative invariance. So, a closed subset A of Rd is negatively invariant. If you start a solution trajectory in A, then for all t less than or equal to t0, this solution trajectory is in A. So, this 'less than or equal to' is connected to this negative word over here. So, now we can talk about what an invariant set is.

So, a closed set A subset of Rd is said to be invariant with respect to an ODE if it is both positively and negatively invariant.

$$A \subseteq R^{d}$$

So, let us quickly summarize what we have done so far. With respect to an ODE, we have introduced the concepts of positive invariance, negative invariance, and invariance. So, invariance, of course, is when you have both positive and negative invariance. So, we now need a few more concepts: one is that of the omega limit set, and the other is that of the alpha limit set.



So, I will define these, and then we will look at a few examples where we will try to better understand what a positive or negative invariant set is, and also what these omega and alpha limit sets are. So, let us first look at the formal definition of an omega limit set. So, given a solution trajectory Xt, T0, X0, its omega limit set. So, notice that an omega limit set is defined with respect to a solution trajectory. So, the omega limit set of a solution trajectory is basically defined to be the intersection of these sets.

$$\omega\left(x_0\right) = \cap_{t\geq t_0}\{X(s),\ s > t\},$$

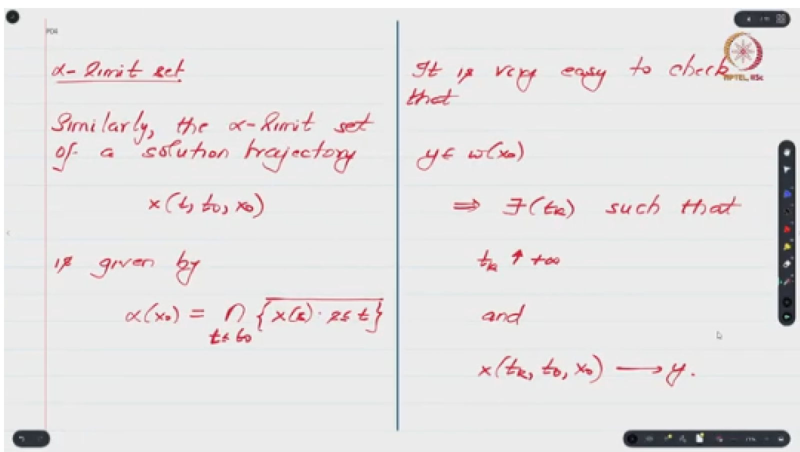
$$where\ X(s)\equiv X\left(s, t_0, x_0\right)$$

So, let us understand what these sets are. So, first of all, the intersection is over T greater than or equal to T0. So, this T0 and this T0 are the same. So, you take T greater than or equal to T0, and this set consists of all values of X of S where S is strictly bigger than T, right? And we use this shorthand X of S to denote the value of the solution trajectory at time S, okay?

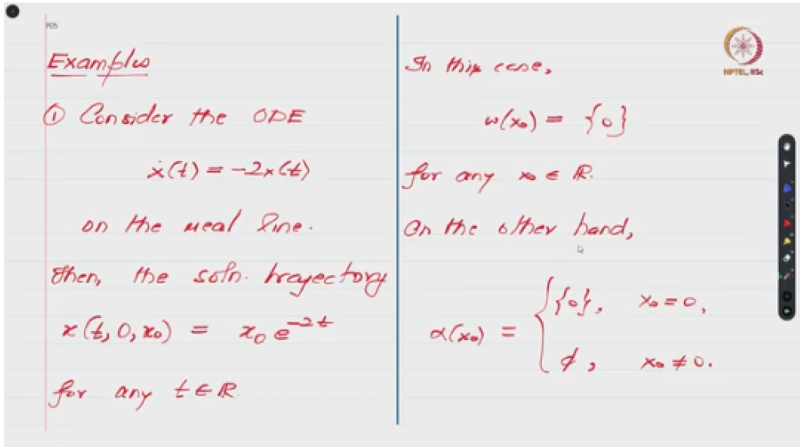
And this bar over here denotes closure. So, let us just understand. So, whatever your solution trajectory is, you take the tail of that solution trajectory. So, I say tail because it is S greater than or equal to T. So, you fix T and look at the solution trajectory from T onwards. You know, collect all those values of X of S. So, each X of S is a vector in R^D.

So, collect this collection of D-dimensional vectors, put them in a set, and take their closure. So, this will be a closed set, right? And now you construct this set for every T which is bigger than or equal to T0, and then you take their intersection. So, whatever you end up with, that is what is referred to as the omega limit set. For those who are familiar with open and closed sets, you can easily see that because you have a closed set here and an arbitrary intersection here, this set will, by definition—I mean, by the fact that you have an intersection of closed sets—this set will also be closed.

Similarly, one can define a concept of an alpha limit set. So, given a solution trajectory X, T0, X0, its alpha limit set is given to be the intersection for T less than or equal to T0, X of S, S. Again, we look at S less than or equal to T and you look at the closure. Is this okay? So, observe that both your alpha limit set and omega limit set are functions of the initialization. It is not a function of T0 because, again, the reason being that we are working with autonomous ODEs, right?



In autonomous ODEs, the precise time where we start does not matter. What matters is where we start and the relative time from this initial time that we have, right? So, now it is very easy to check that, with regards to an omega limit set, if y belongs to omega of x0, then, by this definition of your omega of x0, there exists a sequence tk such that tk is monotonically increasing to infinity and x tk t0 x0. So, this is the value of the solution trajectory at time instance tk. This goes to y.



$$\alpha\left(X_0\right) = \cap_{t \le t_0} \overline{\{X(s) : s \le t\}}$$

$$y \in \omega\left(x_0\right)$$

$$\Rightarrow F\left(t_k\right)$$

$$x\left(t_k, t_0, x_0\right) \rightarrow y$$

So, this condition will be true just from this definition of omega of X naught. Now, let us look at a few examples to better understand what these positive and negative invariant sets are and also what these omega and alpha limits are. So, to begin with, let us consider a very simple ODE. This ODE is x dot of t equals minus 2 times x of t. And this is an ODE on the real line. So, this is simple because one can write explicit solutions to this ODE. For example, x t 0 x naught is x naught times e raised to minus 2 t.

Right. So, we have, just for the sake of simplicity, taken t naught to be 0. So, the initialization is x naught, and we ask: if you start at x naught, what will be the value of the solution trajectory at time t? Then, you know from the basics of ordinary differential equations, one can quickly see that the solution will be x naught times e raised to minus 2t for any t in R, right. So, notice that it is t in R. So, the solution is valid for both t greater than or equal to 0 and t less than or equal to 0.

And in this case, one can see that if you take t going to infinity, this value will keep decreasing, and whatever the value of X naught, this product will go to 0. And because of this reason, one can check that the omega limit of any initialization X naught is actually the set containing 0, right.

$$x(t) =- \ 2x(t)$$

$$x\left(t, 0, x_0\right) = x_0 e^{-2t}$$

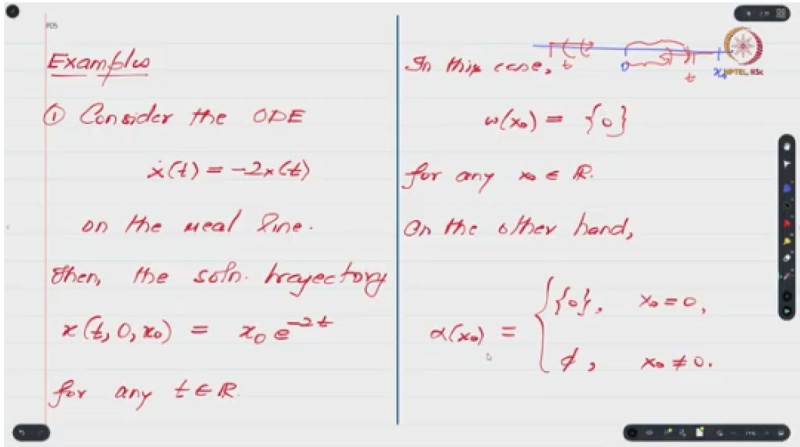
$$\omega\left(x_0\right) = \{0\}$$

So, perhaps I can try to draw a picture over here, right. So, let us say this is your real line, this is your origin, and let us say this is X naught over here.

So, if you start the solution trajectory of X naught, this solution trajectory will continue to move towards the origin. Right. So, let's say, at time t, you are over here. So, the tail of the solution trajectory will be what happens on this side. Is that okay?

And as you increase your t, you know, you will sort of be looking at the tail of the solution trajectory, and if you keep intersecting it, you know, notice that you also have this closure over here, right? So, keeping in mind this closure and the fact that, you know, the solution trajectory keeps moving toward the origin, one can immediately see that the omega limit for this solution trajectory will indeed be 0, right? I mean the set containing 0, and you know, the initialization here does not matter; even if we had started somewhere over here, you know, the solution trajectory then would have moved in this direction. So, again, if we look at the tail end, that is starting from t onward, if you look at the solution trajectory. Right, and take its closure, we can again see that, you know, the tail keeps shrinking, and, you know, if you take their intersection, you will end up with the singleton set containing 0, right.

On the other hand, if you look at alpha of X naught, which is the alpha limit set, so now we have to, you know, traverse in the negative T direction, right. So, if you start at 0, right? So, if you look at this ODE, right? The solution trajectory is given like this. So, if you set X0 to be 0, then this value for any t is actually 0.

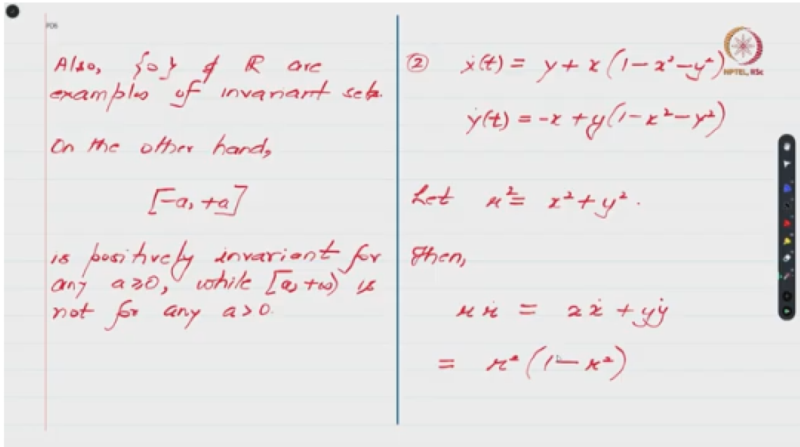


So, whether you go in the positive direction or whether you go in the negative direction, this value is 0. For this special case of X naught equals 0, your alpha limit is actually again the singleton set containing 0. On the other hand, when X naught is not equal to 0, so let us say you start somewhere over here, right?

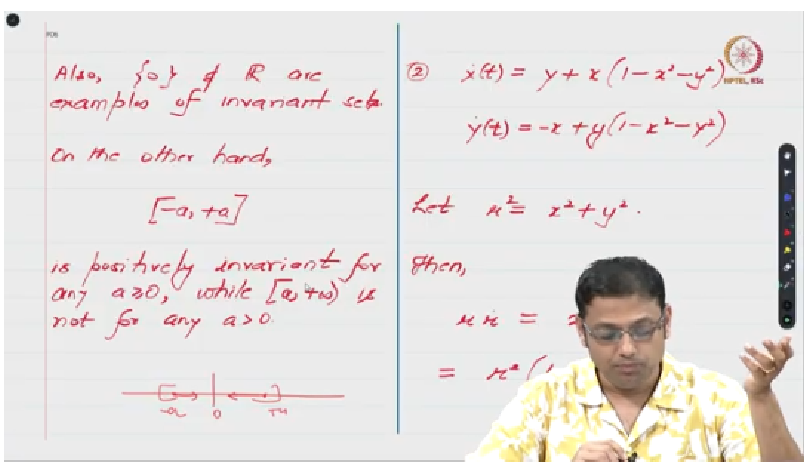
$$\alpha\left(x_0\right) = \{\{0\}, \quad x_0 = 0\ \phi, \quad x_0 \neq 0$$

So now, you know, you have to sort of look at things in this direction time-wise. So, if you look at this and keep taking their intersections, you can see that any point on the real line will eventually be excluded, and because it is getting excluded, one can conclude that the alpha limit set in this case will be the empty set.

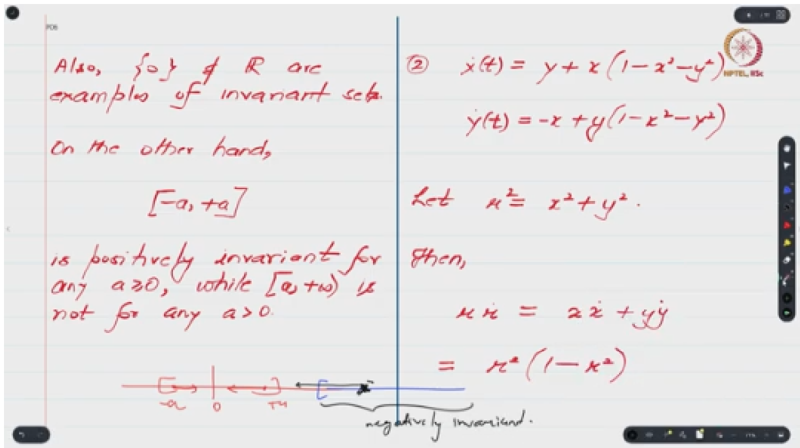
Okay, and now let us look at this particular example. Also, look at examples of invariant sets. From the fact that the solution was X naught times e raised to minus 2t, one can see that if you start at 0, both in positive time and negative time, you will remain at 0. Hence, the set {0}, or the singleton set {0}, is both positively invariant and negatively invariant, and hence it is invariant. This whole real line is also invariant. So, the whole real line is trivially invariant because if you start a solution trajectory of this ODE from any point in the real line, you will of course remain within the real line, both in the positive and negative time direction. Hence, this set is trivially an invariant set. However, if you consider a set of the form [minus a, a], so this is an interval, and



right, such a set is only positively invariant. So, if you start anywhere on this set, let us try to draw this set: this is -a, this is +a, and here is the origin. If you start over here, the solution trajectory will try to move towards the origin this way. Similarly, if you start over here, the solution trajectory will try to move towards the origin in this way. Because of this reason, the solution trajectory will remain within this interval [minus a, plus a] for any t greater than or equal to 0. And hence, one can conclude that this interval is actually positively invariant for any a greater than or equal to 0. On the other hand, if you look at the interval [a, plus infinity), right, so that means some interval of this form. If you start a solution trajectory over here, this solution trajectory will eventually escape this set, and hence this is not positively invariant. On the other hand, if you



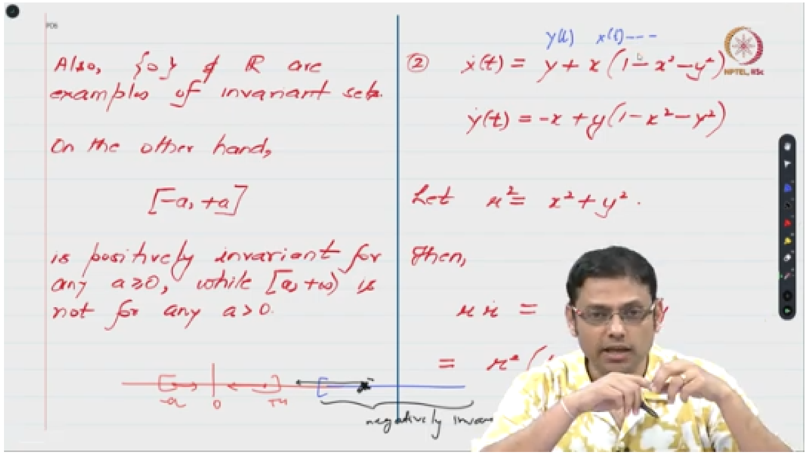
start the solution trajectory here but look at its behavior for negative time, then it will indeed remain over here, and hence this is negatively invariant. So, I hope you are able to see why this is negatively invariant. So, this was in the context of a very simple example. Let us now move on to a slightly more sophisticated example and try to ask or look at these definitions of the positive and negative invariant sets, as well as the alpha and omega limit sets.

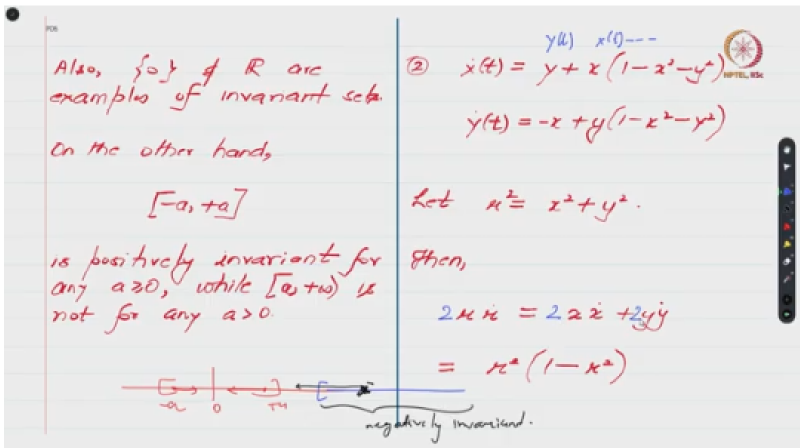


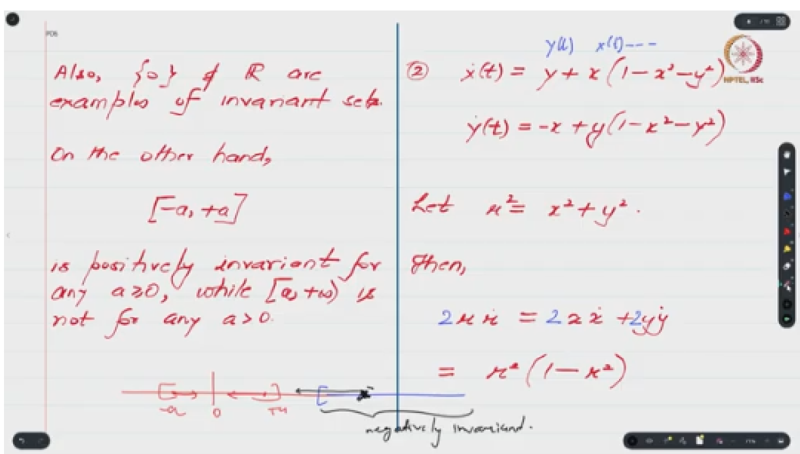
So, let us look at this ODE. So, in this case, the ODE is made up of two variables. So, this is an ODE that lives in R square. So, here, x dot t is y plus t. x times 1 minus x square minus y square, and y dot of t is minus x plus y times 1 minus x square minus y square.

And again, I would like to emphasize that these x and y variables on the right-hand side actually depend on time, and I have suppressed that dependence just for notational simplicity. Ideally, I should have written this as y of (t) and x of (t), and so on. Is this okay? All right.

So, you know, in order to analyze this ODE—in particular, to understand the behavior of the solution trajectories of this ODE—what we will do is we will perform a change of variables. Towards that, we will introduce this variable called r, right, which satisfies the relation r square equal to x square plus y square. Right, and for this, you know, relation, if you take the derivative with respect to time, then on the left-hand side, you will end up with $2r\dot{r}$, right—$2r\dot{r}$, where $\dot{r}$ is basically the derivative of r with respect to, you know, time, right—and, you know, here again, we will end up with 2x times $\dot{x}$ plus 2y times $\dot{y}$. So, if I differentiate this and now if you substitute the value of $\dot{x}$ from here and $\dot{y}$ from here over here, one can see that this expression will eventually result in r square times 1 minus r square.







$$\dot{x}(t) = y + x\left(1 - x^2 - y^2\right)$$

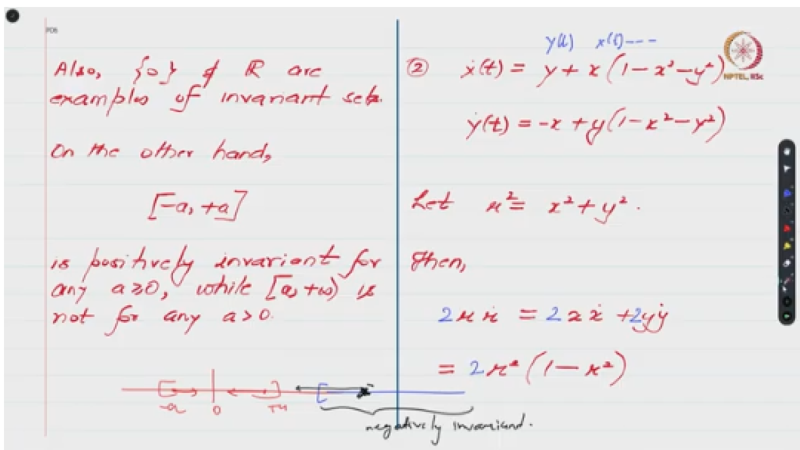
$$\dot{y}(t) = -x + y\left(1 - x^2 - y^2\right)$$

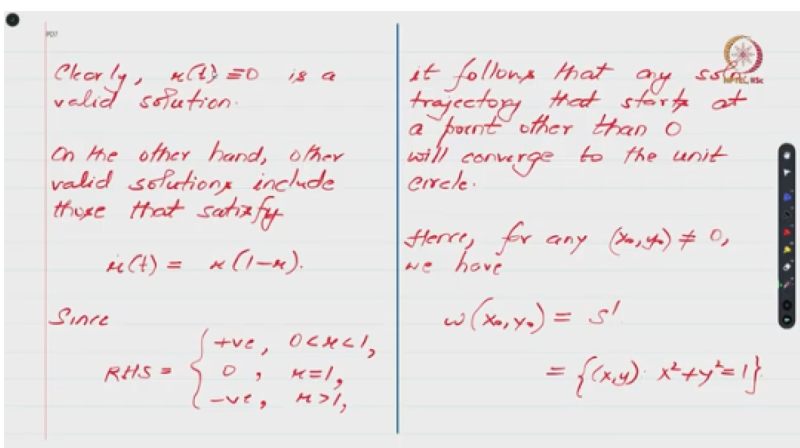
$$r^2 = x^2 + y^2$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$= 2r^2\left(1 - r^2\right)$$

So, from this relation, one can immediately see that, you know, your So, I think I need to be a bit careful here. So, you will end up with 2 here as well. So, in this equation, one can see that if your r is identically 0, then that equation is satisfied. And hence, r dot identically equal to 0 is a valid solution trajectory, meaning you start





at the origin and stay there forever. So, that solution trajectory—I mean, that trajectory—is actually going to satisfy this equation over here. And on the other hand, there are also some other valid solutions for which, you know, your r dot actually satisfies a relation of this form. So, what I have done is I have basically cancelled the r's from here. And hence, I would end up with r dot equals r times 1 minus r, right?

And here, one can see that this right-hand side is having a positive value for r between 0 and 1. equals 0 for r equals 1, and for r bigger than or equal to 1, it takes a negative value, which means that if you look at this transformed variable r's behavior as a function of time, right, when you start the value of r such that it is between 0 and 1, then your r of t value will actually increase. It will increase because r dot of t in that case will be positive. So, which means that, you know, r of t is increasing, right?

So, on the other hand, if you start with an r value which is strictly bigger than 1, then in that case, your r dot of t will be negative. Is that okay? And from this, one can conclude that If you start the solution trajectory of this ODE at a point for which the R value is strictly bigger than 0, then you know you will converge to the place where R equals 1. Now, R equals 1 is basically the set of points where the radius equals 1, that is, the unit circle.

And from this, one can conclude that if you start with any initialization which is not equal to 0, then your omega limit set will actually be the unit circle, which we denote as S1. And what is S1? It is basically the collection of all points x, y such that x squared plus y squared equals 1.

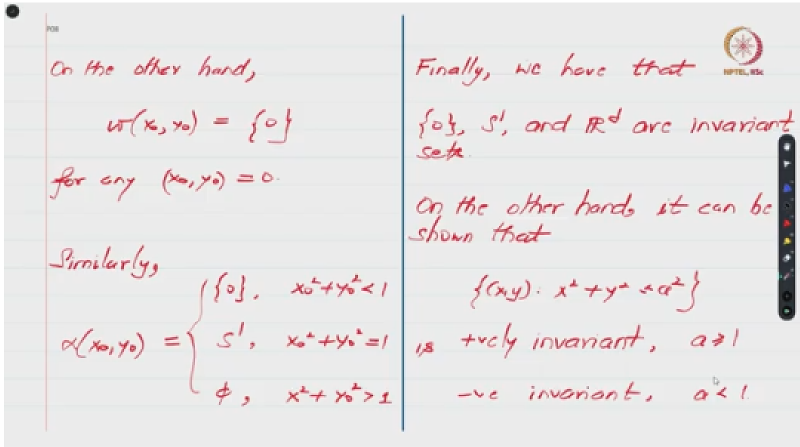
$$\dot{r}(t) = r(1 - r)$$

$$RHS = \{+ ve,\ 0 < r < 1,\ 0, \qquad r = 1,\ - ve, \qquad r > 1$$

$$\omega\left(X_0, Y_0\right) = S^{'}$$

$$= \left\{(X, Y): X^2 + Y^2 = 1\right\}$$

On the other hand, if you start at the origin, if you start at the origin itself, then your omega limit set will actually contain the singleton 0 itself. And with a similar analysis, one can check that the alpha limit set will contain the singleton 0 whenever x squared plus x naught squared plus y naught squared is strictly less than 1.



You know, this alpha limit set equals the unit circle S1 when you start on the circle, right? And it will be empty when your X naught squared plus Y naught squared is actually strictly bigger than 1.

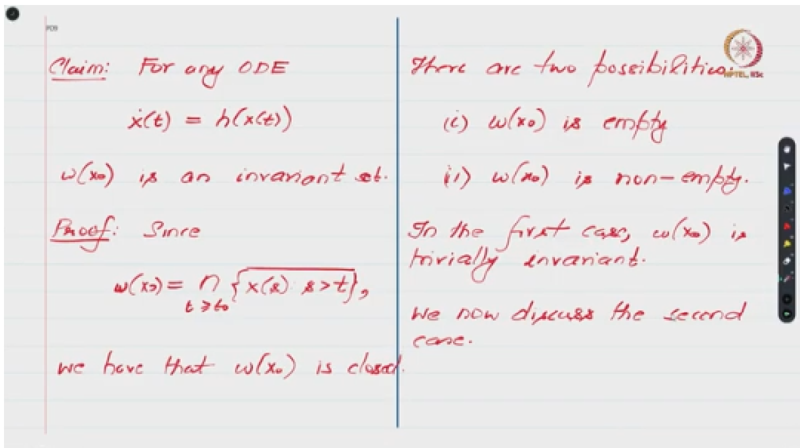
$$\omega\left(x_0, y_0\right) = \{0\}$$

$$\alpha\left(x_0, y_0\right) = \{\{0\}, \quad X_0^2 + Y_0^2 < 1\, S^{'}, \quad X_0^2 + Y_0^2 = 1\, \phi, \quad X_0^2 + Y_0^2 > 1$$

And for this example, one can see that we have three invariant sets. The first is that of the singleton set containing 0, the next is this unit circle, and the third is that of this whole space R2. So, I think I have made a mistake here; this should be R2.

So, these are the different invariant sets for this example, and in the same spirit, one can see that If you take a set of this form, that is the collection of x, y such that x squared plus y squared is less than or equal to a squared, where a is greater than or equal to 1. In this case, this set will be positively invariant because if you start the solution trajectory of that ODE from this set, then the solution trajectory will remain within this set, and hence this is positively invariant. On the other hand, if you take the same set for a strictly less than 1, Then if you start the solution trajectory from this set, it will escape this set so that the solution trajectory can move towards the unit circle, right?

And because of this reason, this set for a strictly less than 1 will not be positively invariant. However, in the reverse direction, the solution trajectory will remain within this set, and hence one can conclude that for a strictly less than 1, this set is indeed negatively invariant, right? We have now come to the end of this class, and you know what remains is to basically discuss this very interesting and useful claim. So, what does this claim say? It says that suppose you have an ODE of the form x dot of t equals h of x of t, and let us say x naught is some initialization, and let omega of x naught be your omega limit set.



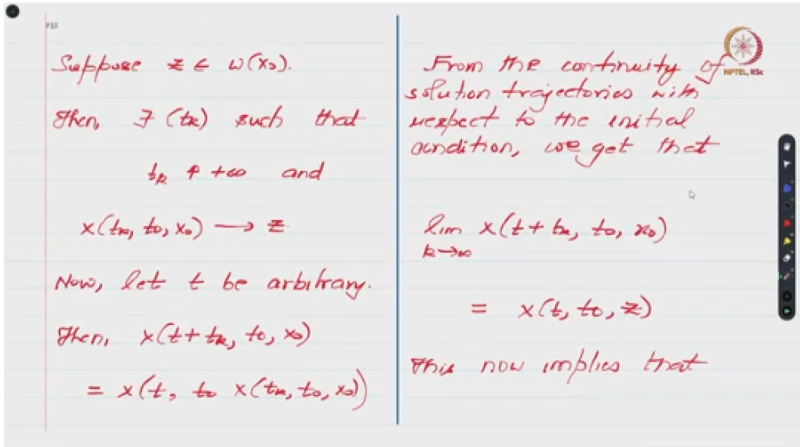
And the claim is that for any X0, its omega limit set must be an invariant set. So, you take X0 and look at its omega limit. So, the omega limit will itself be a set, and that set will be invariant. Meaning, if you start a solution trajectory within that set, whether you look in the positive time direction or the negative time direction, the solution trajectory will remain within omega of X0. So, let us see how we can prove this result.

Actually, this is very easy to prove. So, let us first recall what the definition of omega of X naught is. So, omega of X naught, as you will recall, is the intersection of these closed sets, and because it is the intersection of closed sets, omega of X naught itself is closed.

$$\omega\left(X_0\right) = \cap_{t \geq t_0} \overline{\{X(s): \quad s > t\}}$$

Now, there are two possibilities for omega of X naught. Either omega of X naught is empty, in which case it is trivially an invariant set.

On the other hand, omega of X naught is non-empty, right? So, this case is easy. So, we will only focus on the case where omega of X naught is non-empty, right? So, now let us see why omega of X naught has to be invariant. Towards that, let us consider an element in omega of S naught.

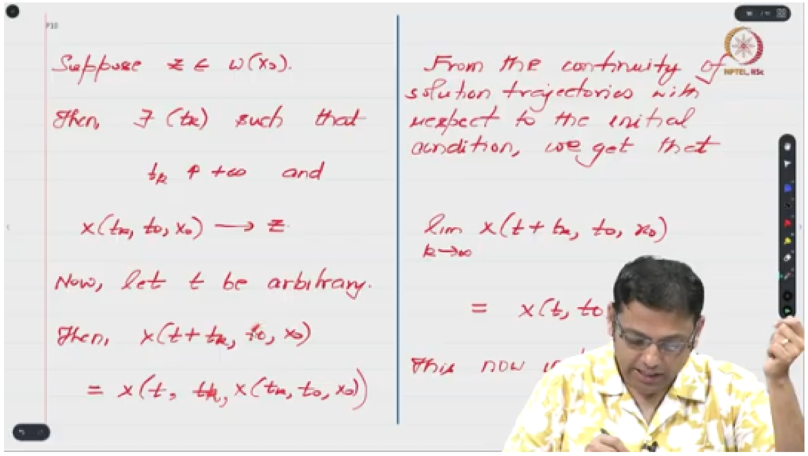


So, now let us first understand what our goal is. Our goal is to show that if you start a solution trajectory at Z, right, and then if you look at the behavior of the solution trajectory both in positive time and in negative time, we want to show that the solution trajectory will lie in omega of X naught itself. So, now from the fact that Z belongs to omega of X naught, one can look at the definition of omega of X naught and conclude that there exists a sequence of time instances Tk such that Tk increases to infinity and the value of the solution trajectory starting from X naught at time Tk. So, these are some vectors. So, this sequence of vectors actually converges to Z.

So, this is, you know, from the fact that Z belongs to omega of X naught, right? And now, recall our goal. Our goal is to show that, you know, if you start the solution trajectory from here and look at T time units ahead or behind, right? And we would like to ask whether the solution trajectory at this time instance lies in omega of X naught or not. So, towards that, let us consider some T in R which is arbitrary, and let us look at this value.

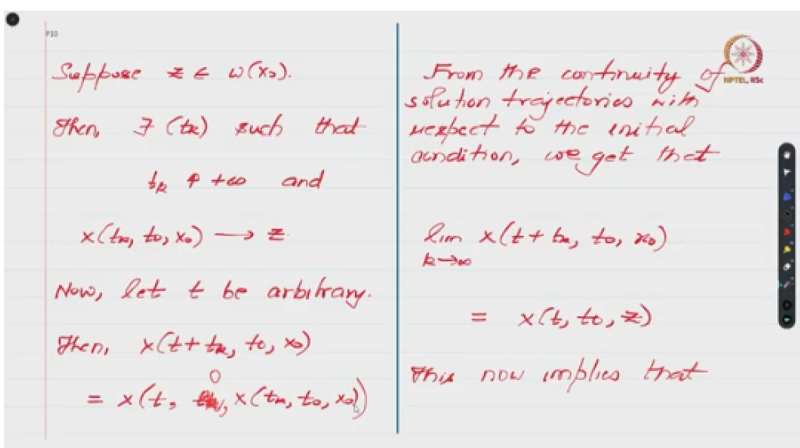
So, we take T plus Tk, T0, X0. So, T is fixed, and we are only going to change this Tk value. So, my claim is that this value is actually equal to this value. So, let us understand the difference between this and this. This solution trajectory at time t0 starts from x0.

And then we are asking where this solution trajectory would be at time t plus tk. On the other hand, this expression over here says that at time t0, At time tk—sorry, there is a typo here. So, let us say at time tk, you are at x tk, t0, x0, right? So, this is the solution trajectory which at t0 was at x0, right?



And we ask, where has it reached at time tk, right? And this is the initialization time now. And then we ask, if you start over here, after t units of time, where would you be? And from the uniqueness of solution trajectories, one can see that this expression and this expression must be one and the same. However, the difference between this and this is that—just one minute, I just want to think of one thing.

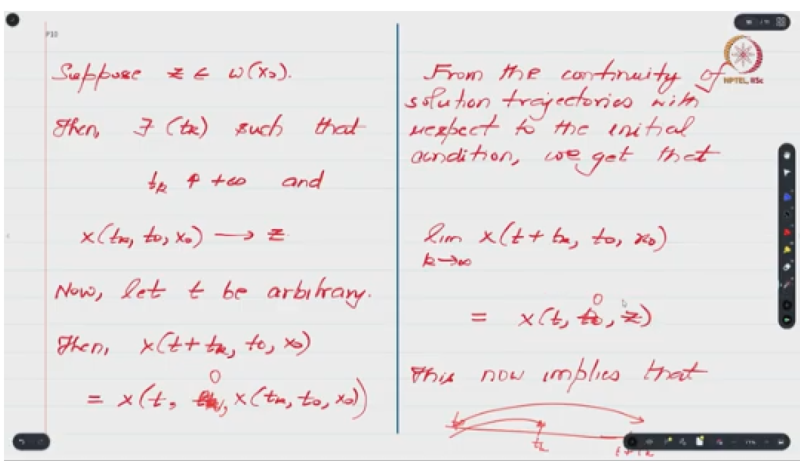
So, I think what I should instead say is, this is actually one. So, this is actually 0. So, you start at 0 at this place, and then you ask, after t units of time, where would you be? So, as I told you, since we are working with autonomous ODEs, the relative gap is what matters. So, if at



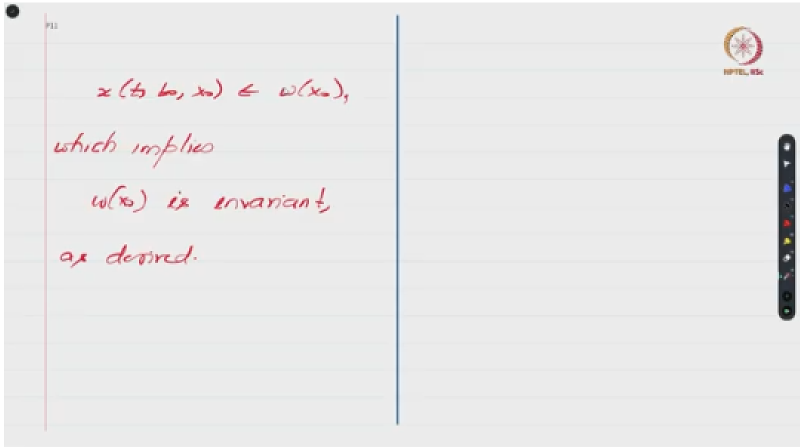
So, if you start over here, right, after T units of time, where would you be? And from the uniqueness of solution trajectories, one can check that these two expressions are one and the same, right? The nice thing about this right-hand side is that the dependence of Tk is only over here. And now, invoking the continuity of solution trajectories with respect to the initial point, one can conclude that as K goes to infinity, the initialization condition actually goes to Z. Hence, if I take the limit as K tends to infinity over here, this expression should go to this expression. This is from the continuity of solution trajectories with respect to the initial condition.

So, let me just make sure I am doing it correctly. So, this is t naught. So, here you have t naught, here you have tk, and here you have t plus tk. So, if you start from t naught and then you are asking where you are. So, that is equivalent to saying you reach here and then you ask where you are at time t plus tk, which is equivalent to this.

So, this is correct. And here, what I should again write is, you know, I should again write it as xt 0 z. Very good. So, this is okay to me. So, this limit is the limit of this, and this initialization actually converges to z. Hence, this limit actually converges to this. Right now, because this limit converges to this, if you again look at, you know, the definition of your omega limit set, one can see that because this limit converges to this vector over here, right, one can conclude that this expression



must also be in omega of X naught. This just follows from the definition of omega of X naught. Recall that omega of X naught has these intersections, right, and the fact that your limit actually converges to this value would tell us that this expression also lies in omega of X naught. So, let us now try to understand what we have proved. We have said that if Z belongs to omega of X naught, then for any T



This expression will also belong to omega of X naught. And since this T is arbitrary, one can conclude that this expression belonging to X naught implies that the solution trajectory started at Z lies in omega of X naught both for positive time and negative time, which then helps us conclude that this omega of X naught indeed is invariant. So, this brings us to the end of the class. Let me quickly summarize what we have done. In this class, we introduced some terms like positive invariance, negative invariance, omega limits, and alpha limits, and we saw some examples of ODEs. We saw examples of these different sets and eventually proved that the omega limit

of an ODE—I mean, the omega limit of any initial condition—is actually an invariant set, okay. So, in the next class, we will understand a bit more about the nature of some limit sets like this, right. Until then, bye. Thank you.