NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Discrete Mathematics
Graph Theory – 2

Dirac's Theorem

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Let us now look at this celebrated theorem called the Dirac's theorem, it states whenever you have this condition that for all the vertices it is true that degree of a vertex is greater than or equal to n/2, then you can always find a Hamilton cycle,
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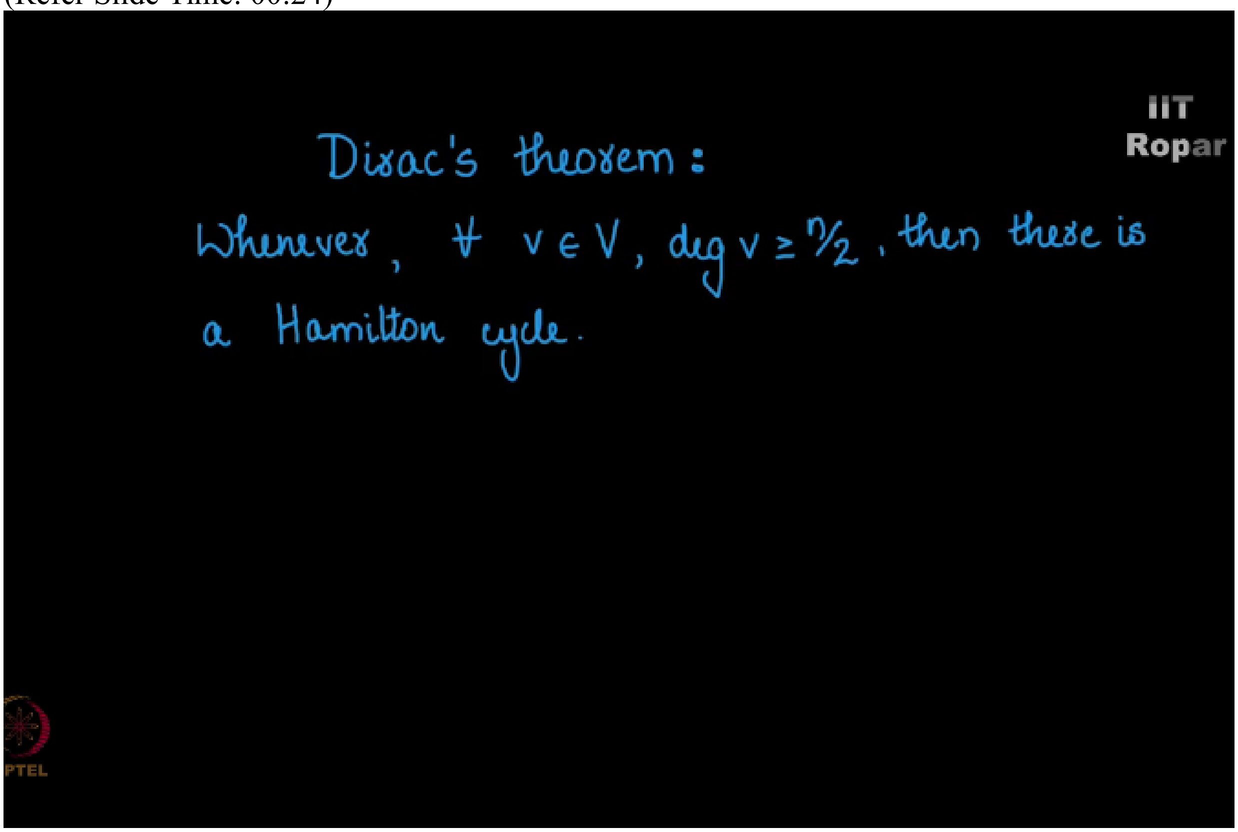

a cycle that goes through all the vertices, but proof of this is going to be slightly non-trivial, so we better concentrate, okay.

My first point is the graph is connected if this be the case,
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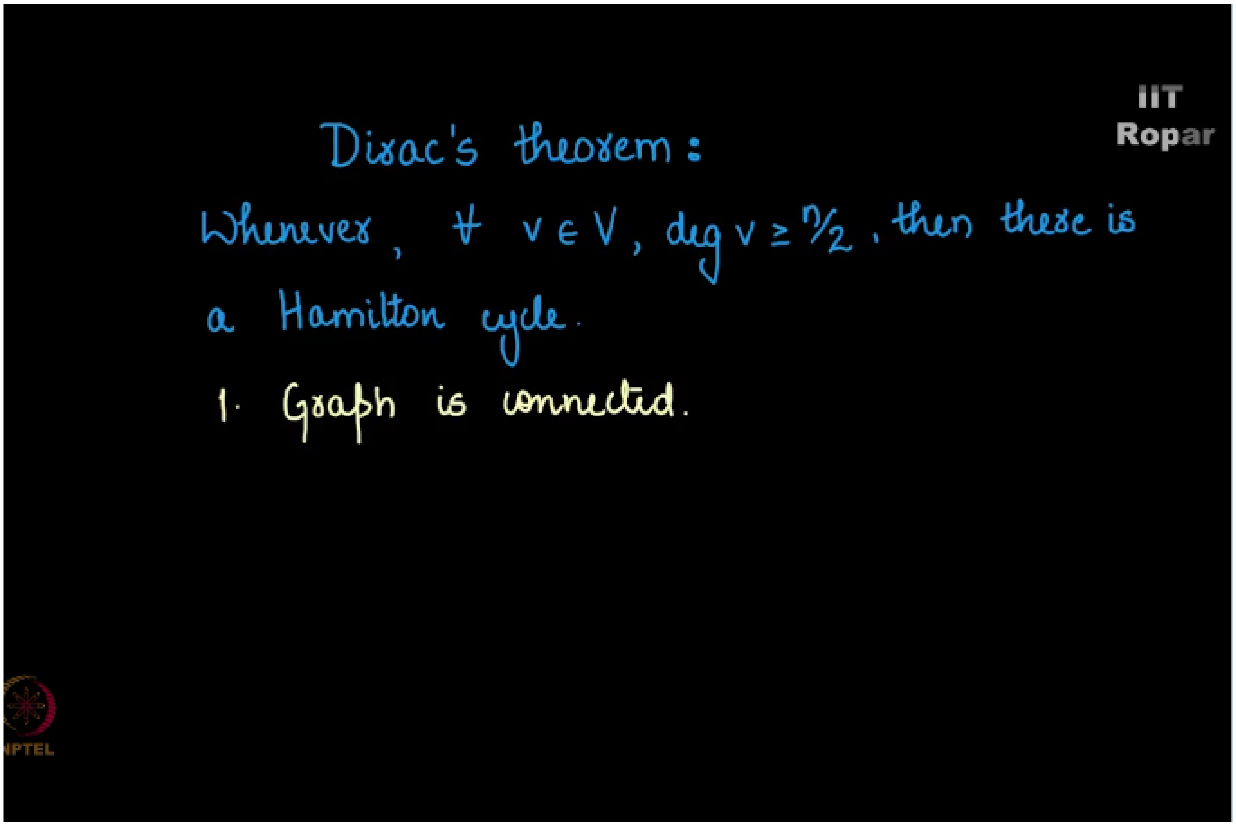

when every degree of every node is greater than or equal to n/2, why is that? We already saw, so I'm not going to go into it once again, I'm going to assume the result in case you don't understand if you want to go back and watch the previous videos, so I know the graph is connected.

What I'll now do is I'll take a path, let's say U1, U2 up to UK, which is the longest path in this graph, the longest possible path
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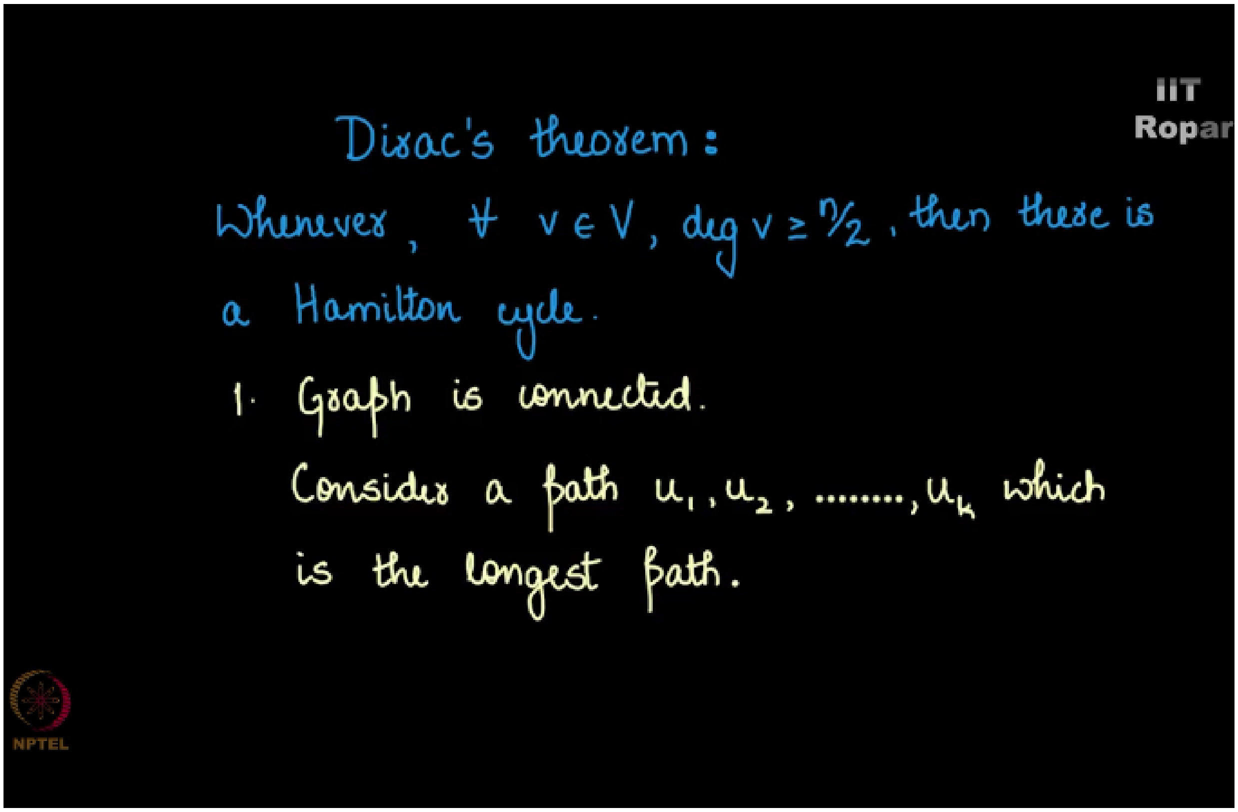

whatever that is, now a Hamilton cycle is something that goes through all the vertices, okay, which means if you take the Hamilton cycle in the graph that itself will give you a path that is the maximum possible which is something that covers all the nodes, you cannot have a path bigger than that, so this longest path will actually be such that your K will be equal to N, think about it,

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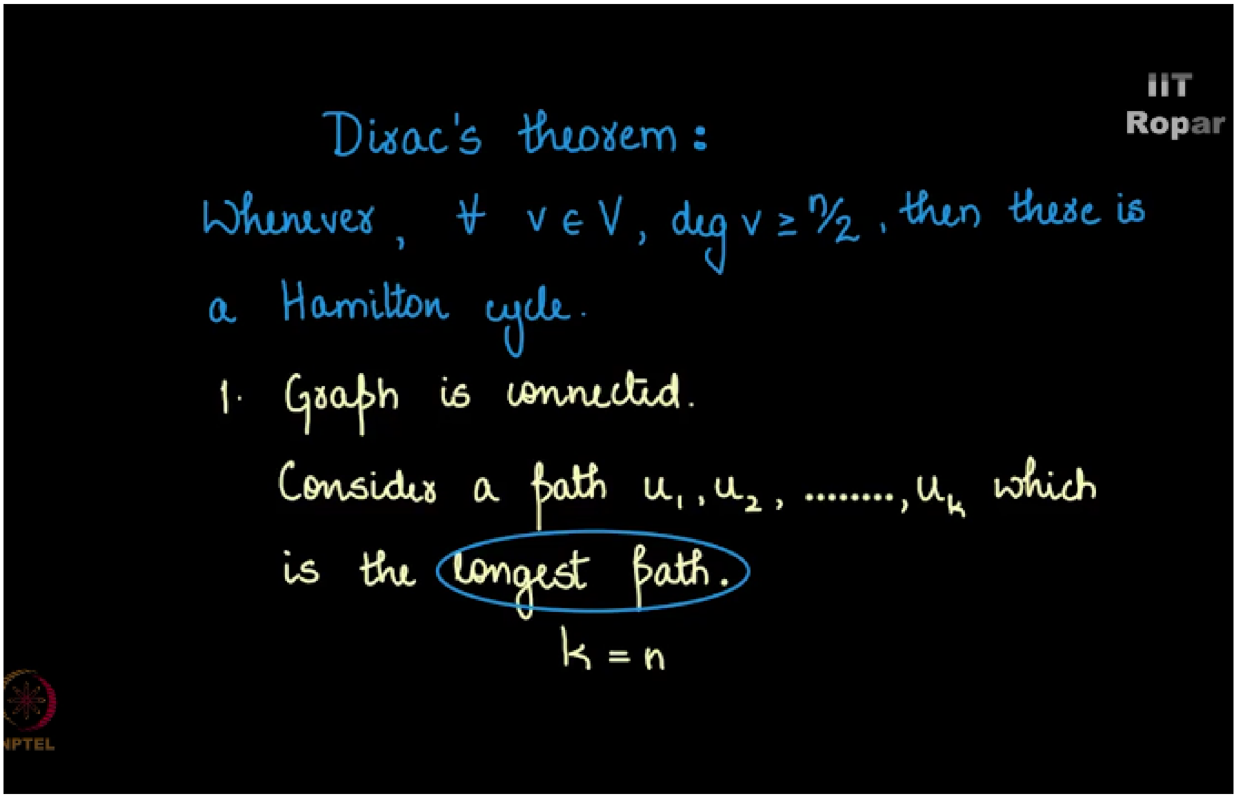

if there is a Hamilton cycle definitely your longest path in the graph will be such that, K = N, but this may not be true if the Hamilton cycle does not exist, in that case I will take the longest path.

If this longest path is such that K is less than N I will continue my argument just watch, so I'll assume that K is less than N, okay, now look at this,

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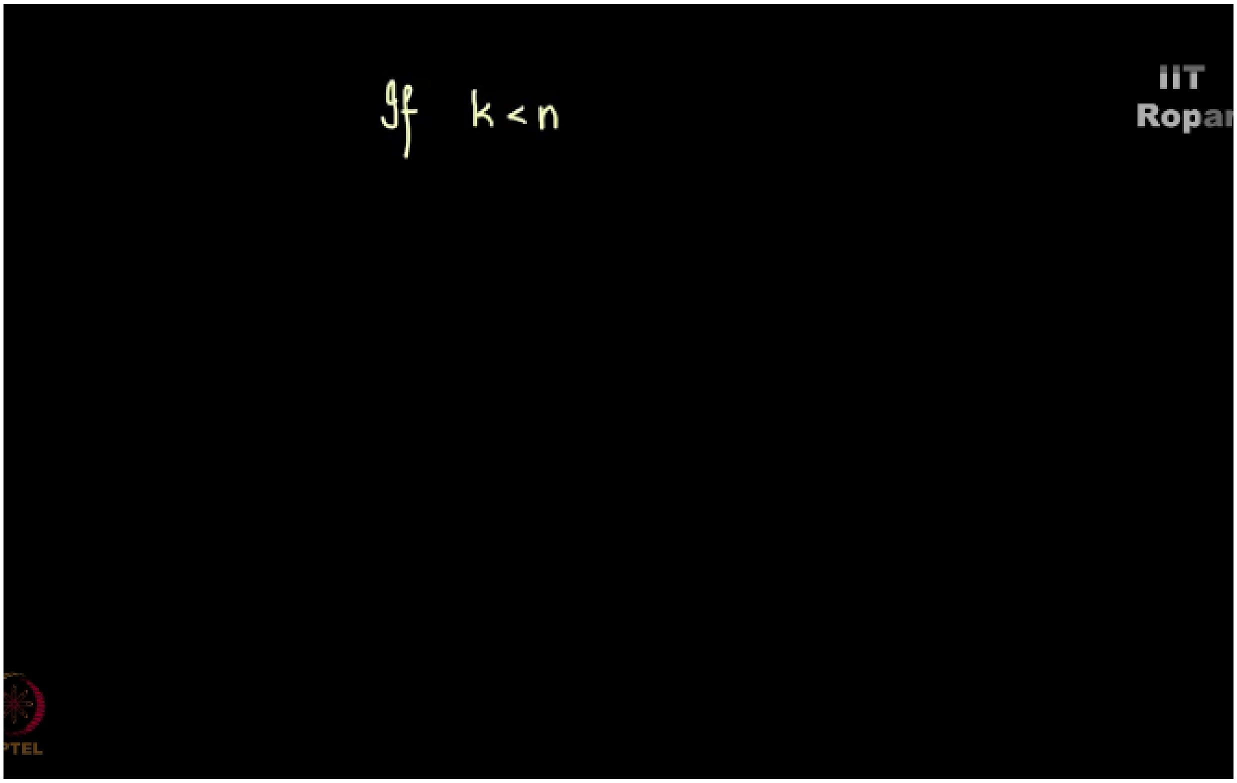

let me give you a small analogy, instead of calling it KL call it 100 so that it's easy on our minds, U1, U2 up to U100, so U1 is me Sudarshan, U2 is let's say you, let's say your name is Ramya, so Sudarshan has some 50 friends or more, correct, because degree of every single person here is greater than or equal to n/2, n is 100 and N is greater than 100 in fact, so you are taking vertices at most 100, U100 means I should be friends with people more than 50 here, okay, greater than or equal to 50, so what we do is I'll go up to U80 for clarity sake, there are 100 nodes, right, and I'll take the longest path let say the longest path is of length, whatever it has 80 vertices, now U1 is Sudarshan, U80 is let's say Ramya, which is you, and U1 which is me I have 50 or more friends, okay, 50 or more friends, and you have 50 or more friends, okay, and as you can see just look at U1, and just look at U80,

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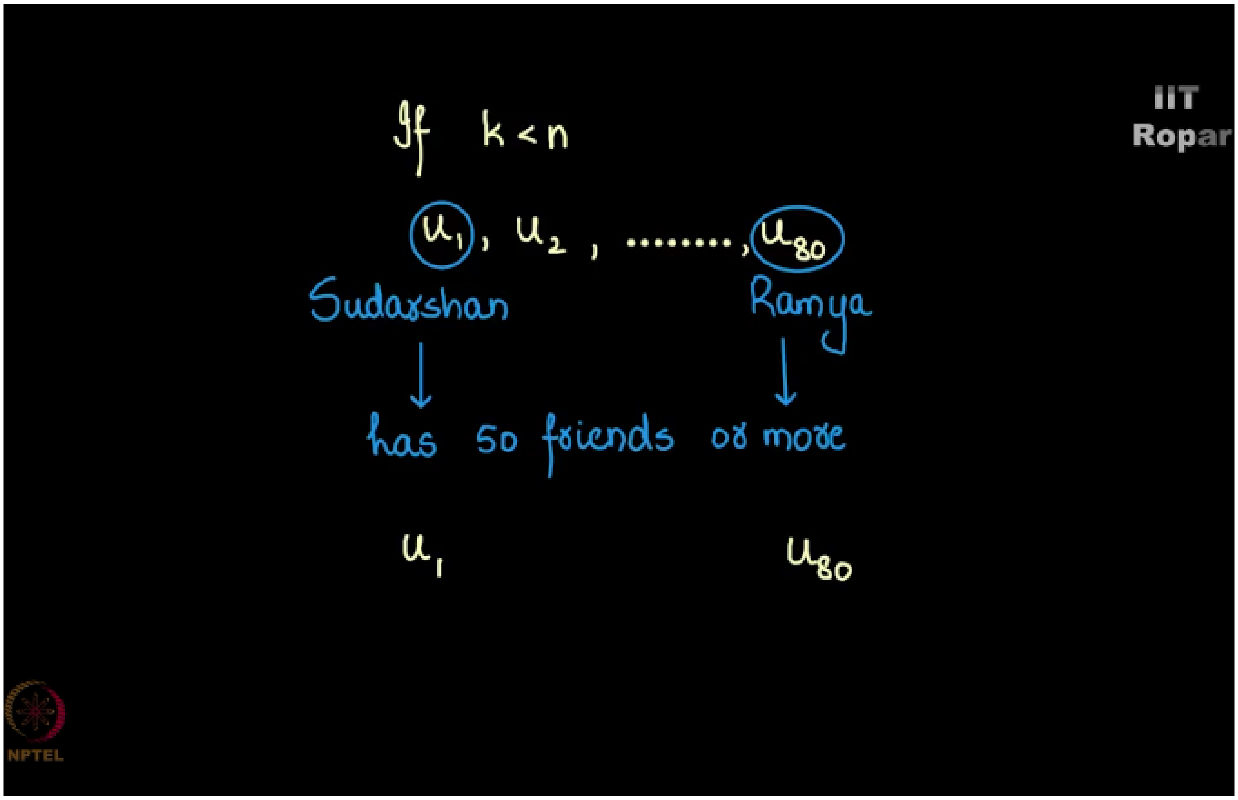

my question is do we see a structure like this here, where let me call this X, let me call this Y where U1 is adjacent to Y, and X is adjacent to U80,

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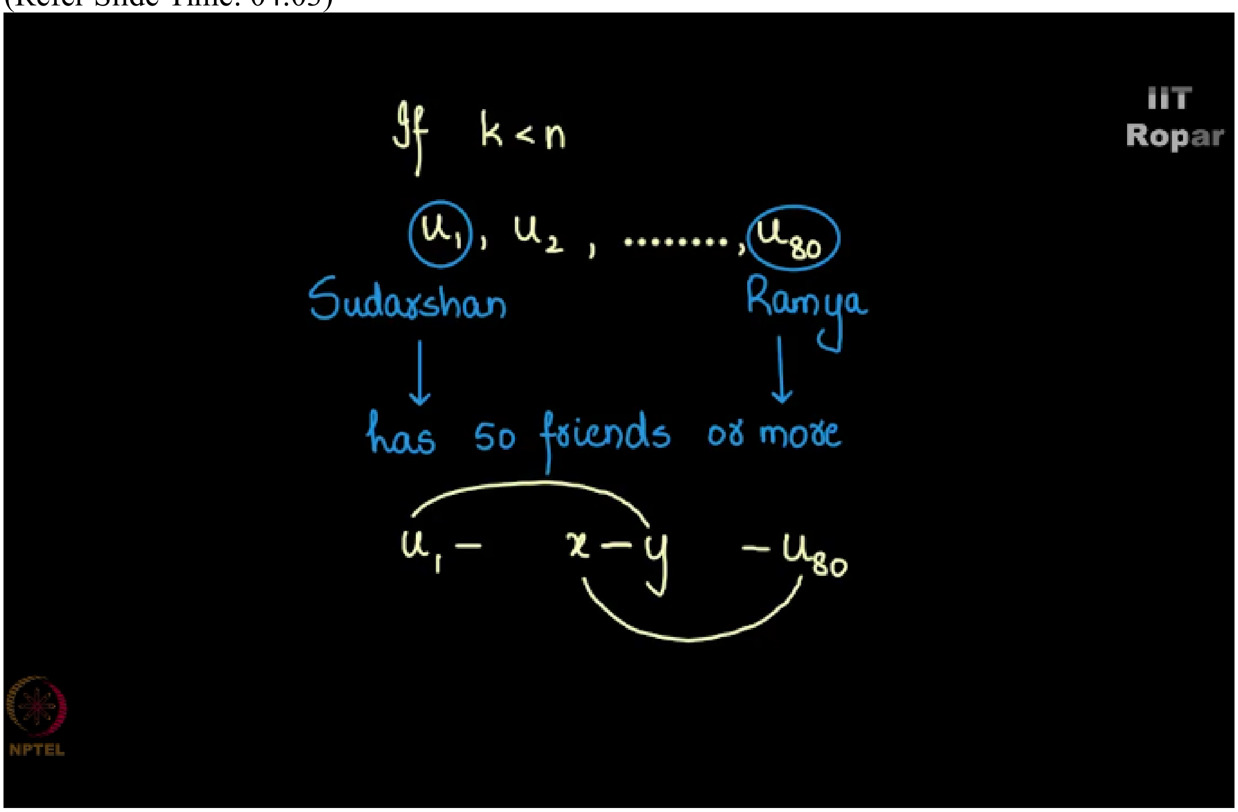

I just now told you a while back that this may or may not be possible, but I say this is always possible here, why?

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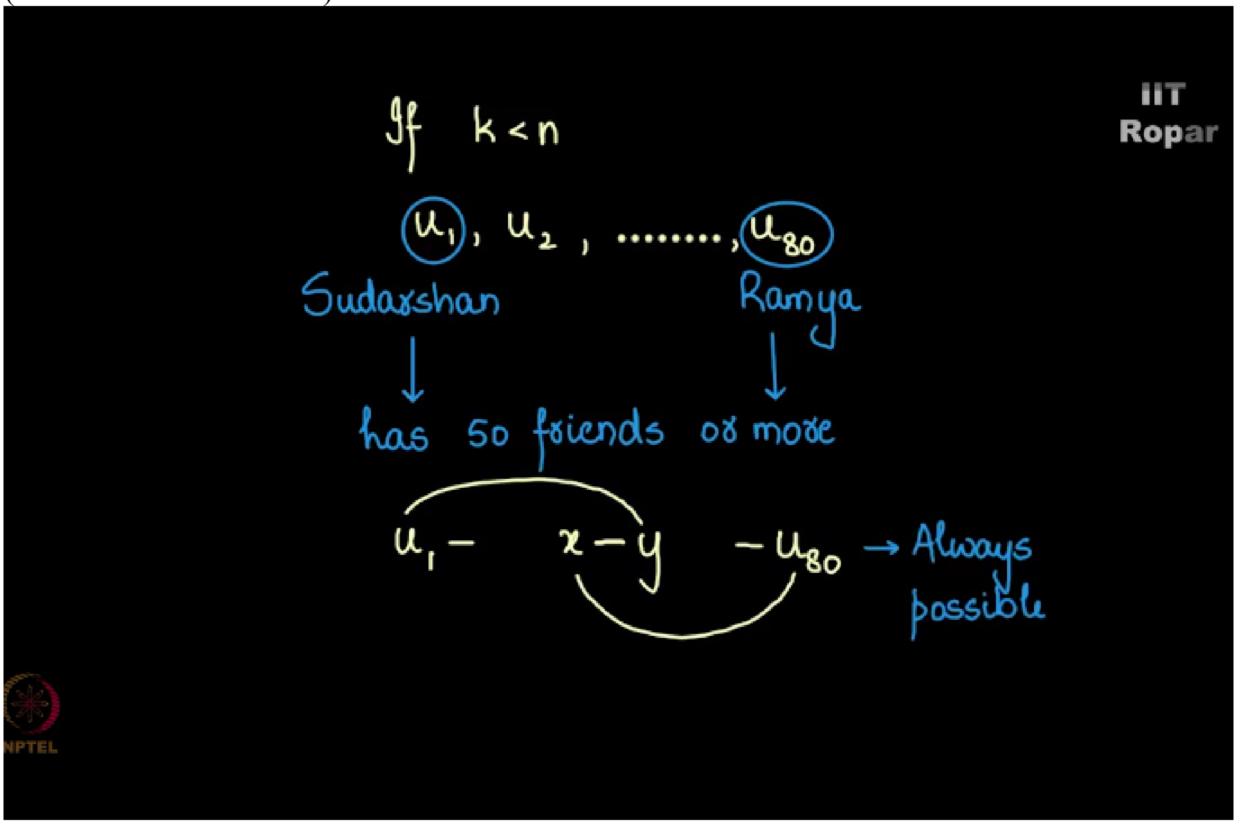

That is because for every vertex that U1 is adjacent to, the predecessor of that is not adjacent to U80, then you will have 50 friends of U1, and none of the predecessors are friends with U80, and the number of nodes here will be greater than 100, that is not true,

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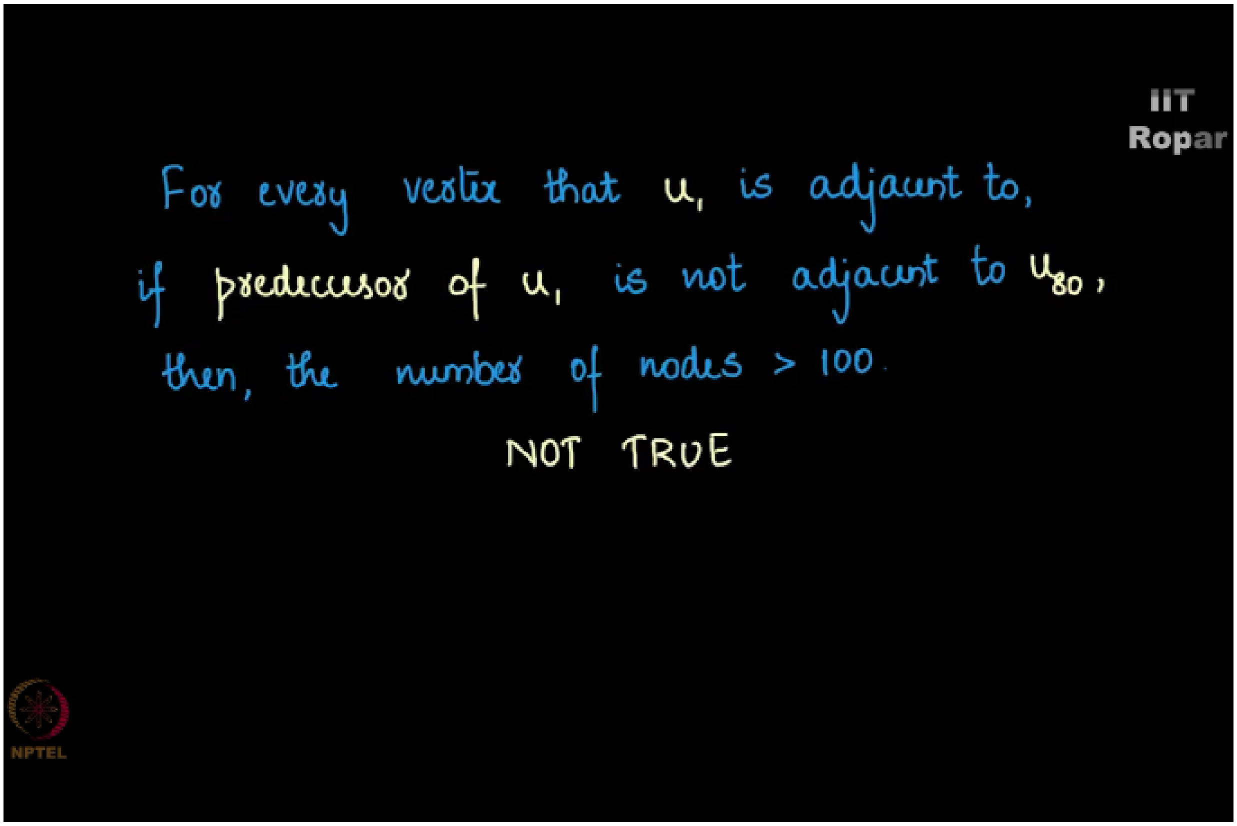

there were at most 80 vertices here, you see I mean there is indeed 80 vertices here, so and also there is one thing U1 cannot be adjacent to a vertex outside these U1, U2 up to U80, why is that?

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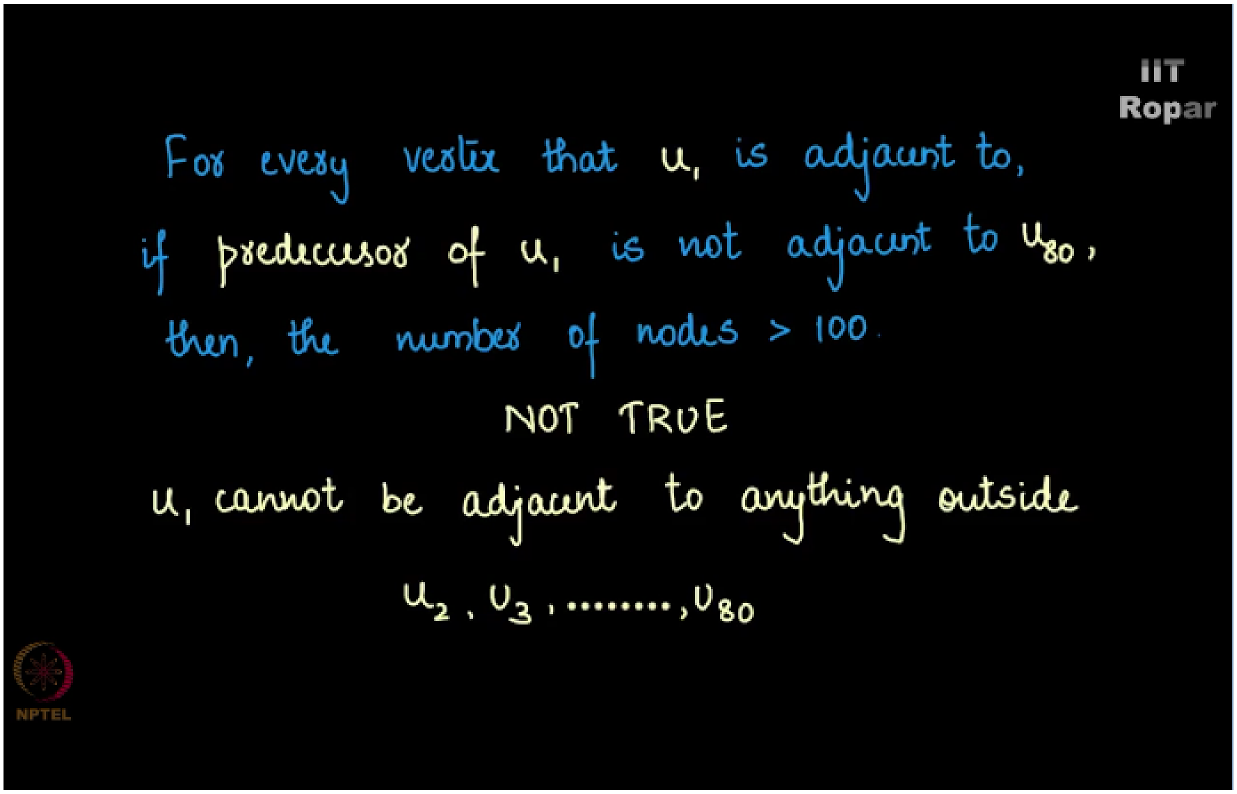

That is because we considered this to be the longest path in case U1 was adjacent to some U dash outside this path, then U1 to U80 will not be the longest because you can always extend U1 to U dash and make it one length longer, correct, so all in all I'm trying to say that a structure like this will definitely be there, now this is a really point of hurdle for many of you in this big theorem,

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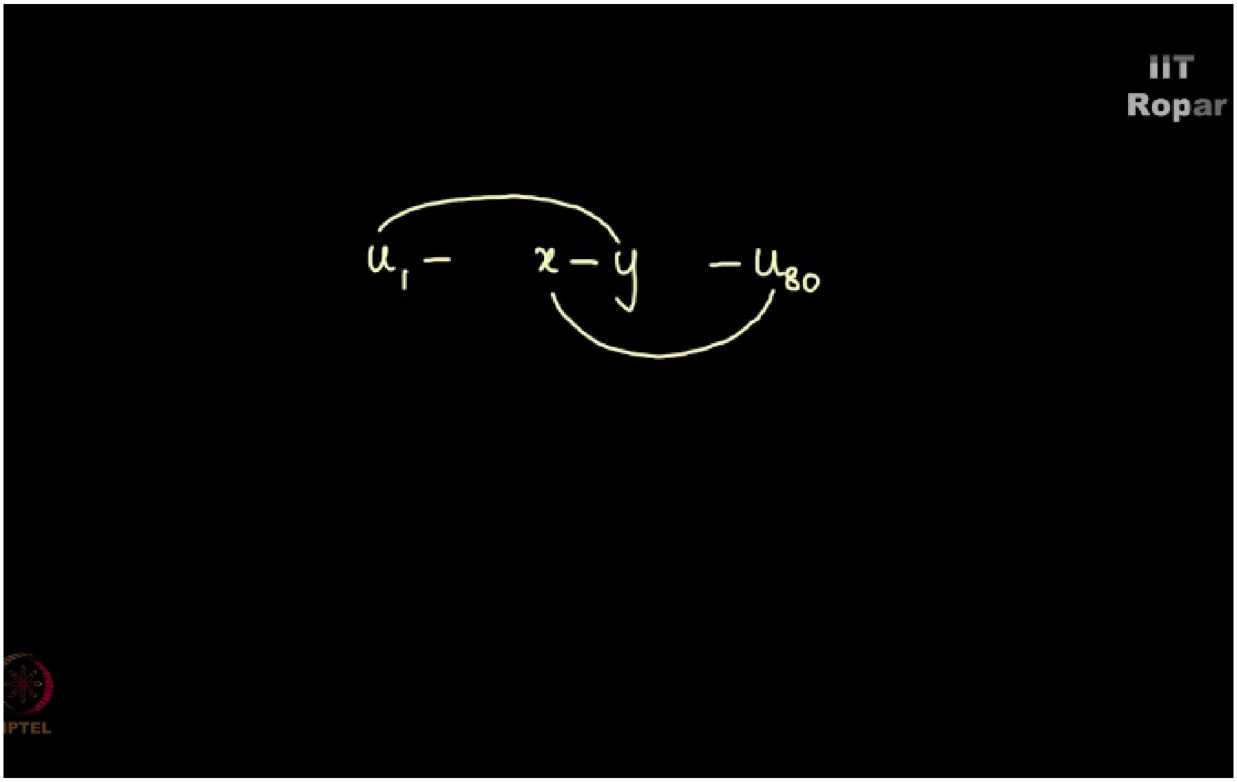

this is a major theorem and one thing that people don't understand is this very step, how do you find such a pattern, right, you will always find the pattern for some X, Y here on the path, U1 will be adjacent to Y, and X will be adjacent to U80, if that is not true then for every vertex adjacent to U1, its predecessor is not adjacent to U80 and U1 is adjacent to 50 such nodes,
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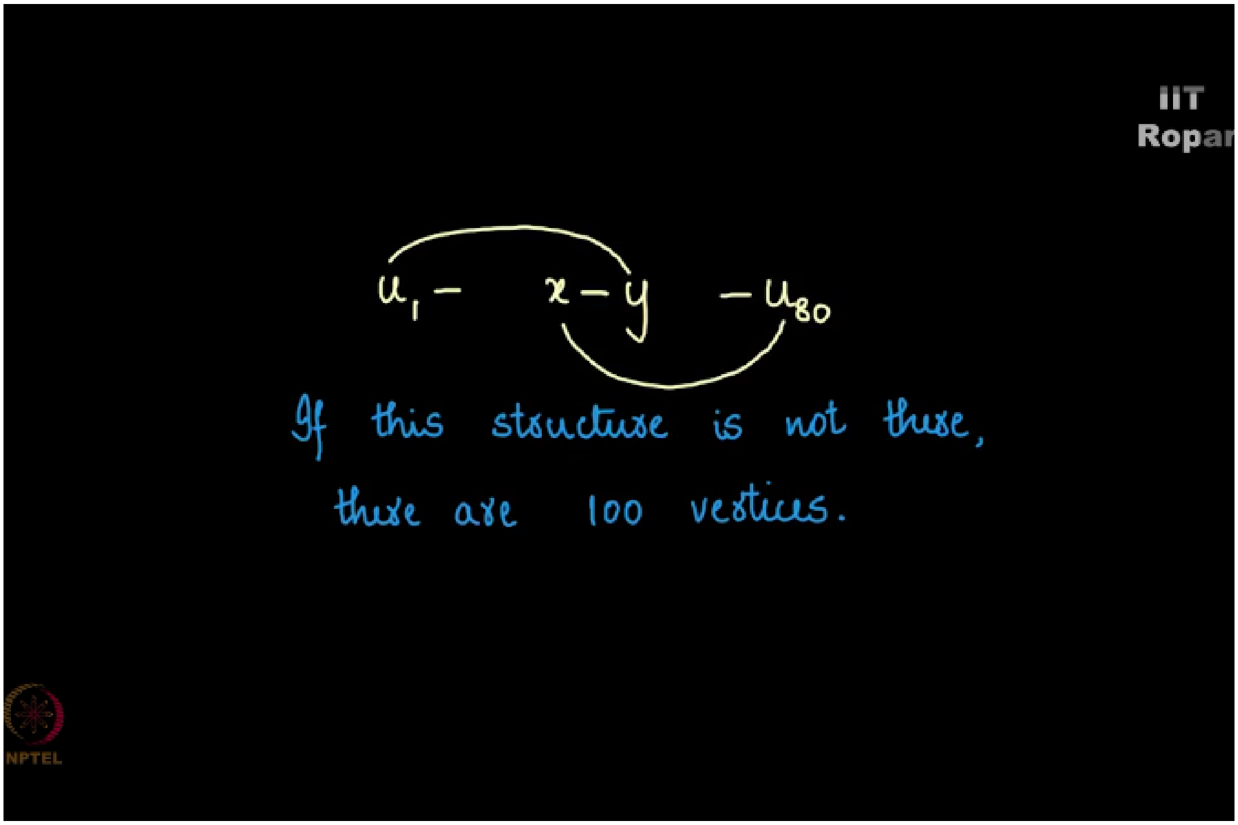

and for 50 such nodes the predecessors are not adjacent, which means 50 + 50 there are 100 vertices here in total, which is contradictory because you only have at most 80 vertices here, so which means there is a structure like this and when you have a structure like this simply remove X, Y you will get a cycle, a cycle on 80 vertices,

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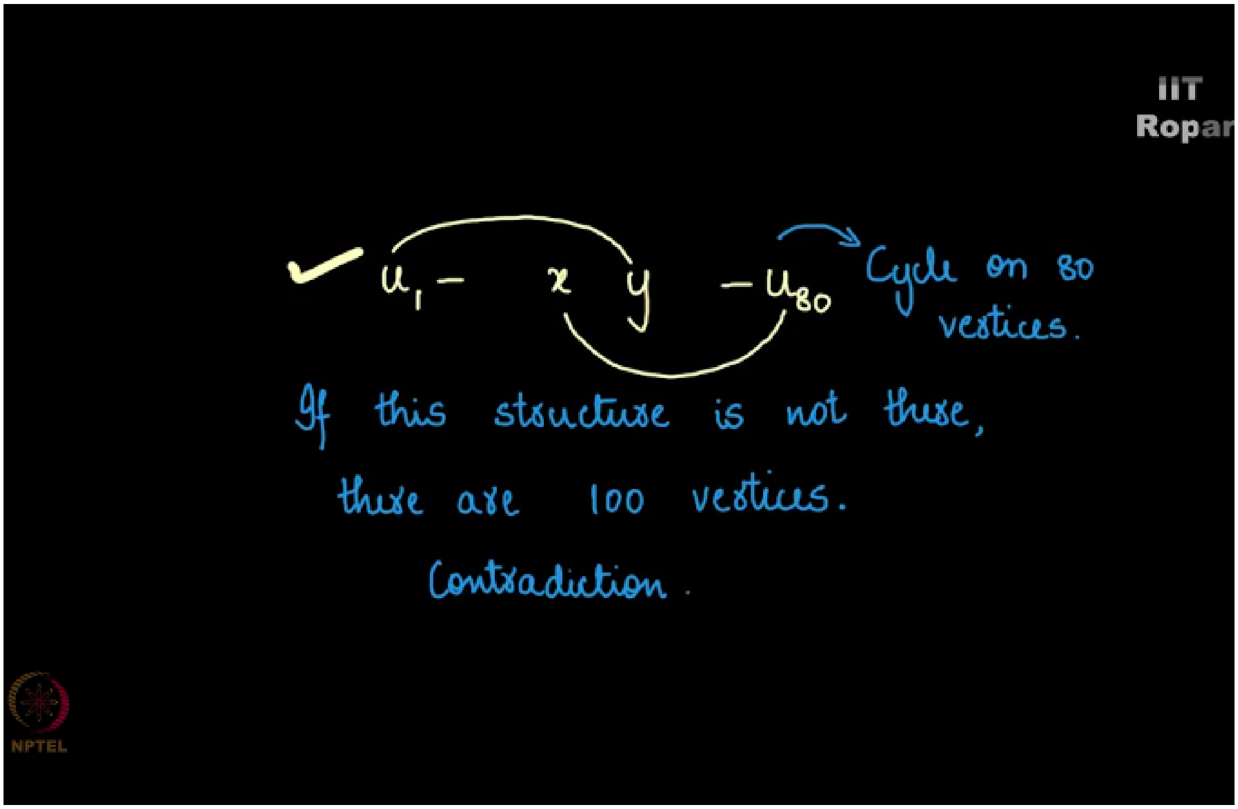

if the graph is on 100 vertices, correct, but this path was on 80 vertices which is now a cycle on this very graph that you obtained, correct, all I'm saying is the graph G, the bigger graph G you take a path from U1 to U80 but you have found a cycle on this very path, and when you find the cycle in a given graph on let say 80 vertices you can always pick any node outside the cycle, there is always a path to the cycle,

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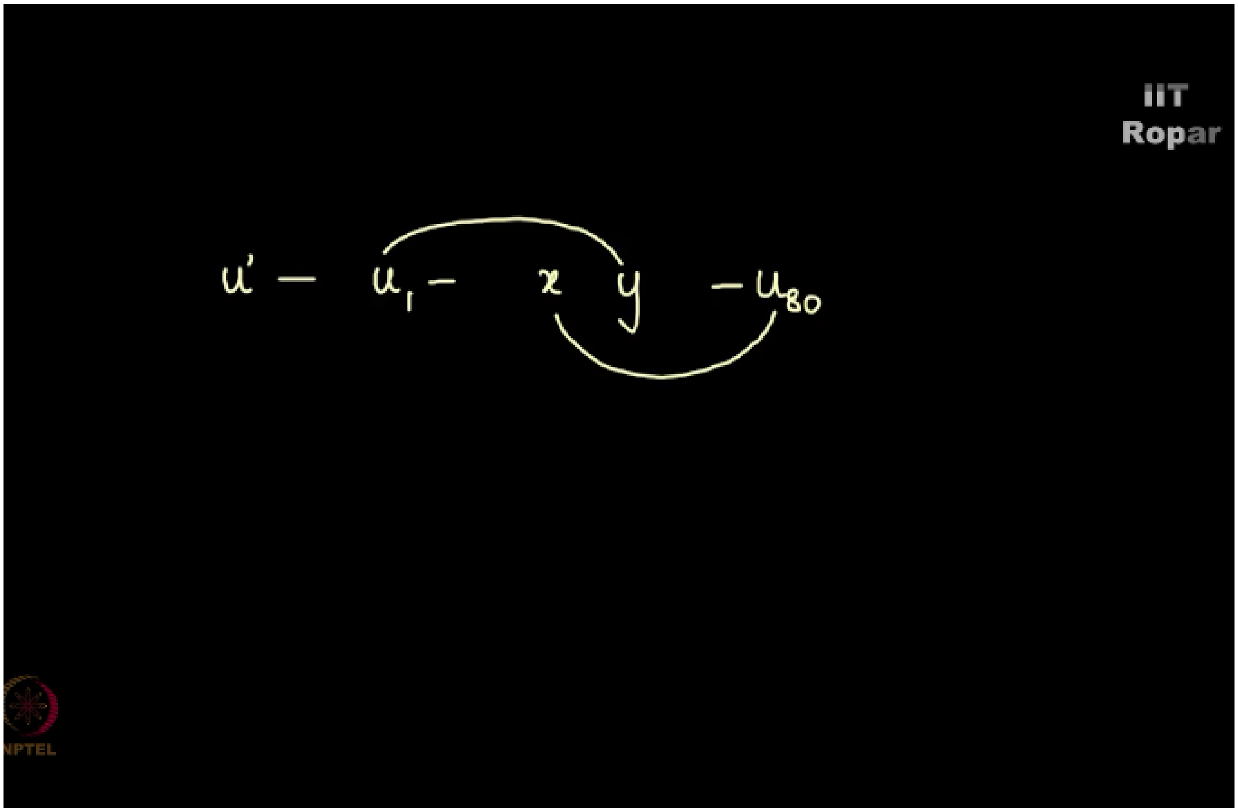

because it is connected and then you can remove this particular edge, you get a bigger path, correct, a bigger path than this path with 80 vertices boom, that's a contradiction to the fact that this particular path was the longest,

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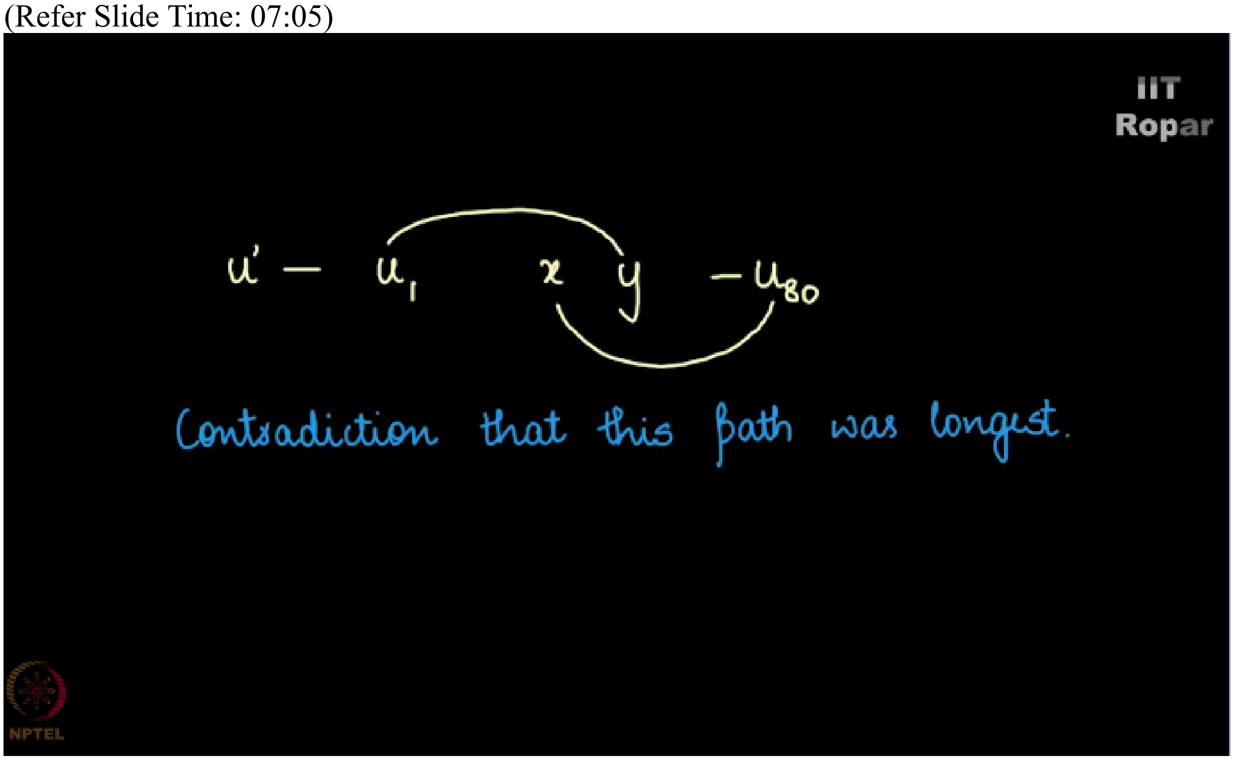

correct, if you take a path of length 80, not just 80 you take any path of the maximum possible length in this graph, right, I'm contradicting you by showing you that this is a longer path here, whenever this condition happens, degree of every vertex is greater than or equal to n/2,
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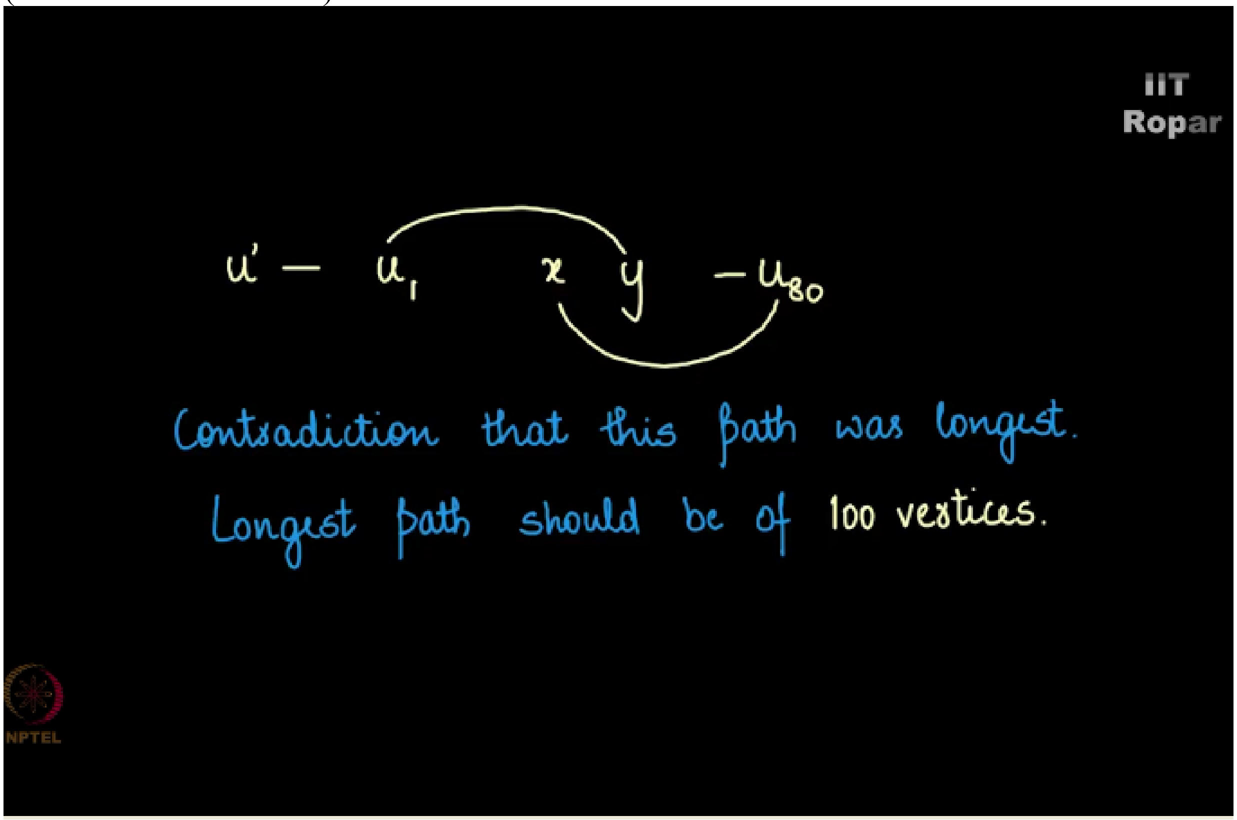

which means, you cannot have a longest path that is less than 100 vertices, the longest path here should be of 100 vertices length, a word of advice as and always, a theorem which has a proof this longish will not be so clear to you on your first attempt of understanding it, you must watch this video many times, think about the proof and only then will it be clear to you.

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