Computer Vision

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Lecture – 21
Stereo Geometry Part – VI

We are discussing about fundamental matrix of a Stereo Geometry and we have seen how fundamental matrix plays a very distinctive role in characterizing its geometry. And in the last lecture, we discussed how fundamental matrices could be estimated given the observations of corresponding pairs of points and also some parametric forms of representation of fundamental matrix which also sometimes helps us in formulating the estimation problem.

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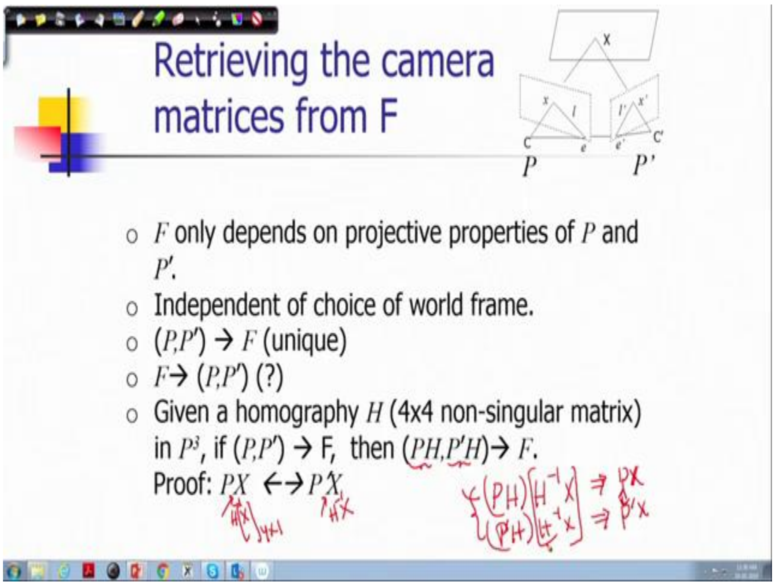

Now, in this lecture, we will be discussing that how we can retrieve the camera matrices from fundamental matrix F. So, note that in our previous problems, we have not considered estimation of camera matrices. If we know the camera matrices, then computation of fundamental matrix is very direct; it's very simple. We will see here the reverse is not the same.

So, let us consider these aspects that if that is a fundamental matrix, it only depends on the projective properties of P and P'. So, given a pair of P and P', you get an unique F and the interesting part is that it is independent of choice of world frame means world coordinate system. That means, if I have a stereo setup and I can move this set up by keeping its relative orientation the same; relative orientation of the image access the same.

I can move this stereo setup in at any point of the world frame whatever may be my world coordinate convention here. Still I should have fixed world coordinate system and then, I can move the stereo system at any point, still we will get the same fundamental matrix which is not true for projection matrix. If my world coordinates also move and based on that the corresponding points of image coordinate and world coordinate points will change. Because world coordinate have been transformed then your projection matrix will also get transformed, but this is not true for fundamental matrix. It is solely dependent on the image coordinates.

So, if you have same pairs of image coordinates for the corresponding pairs of points, we will get the same fundamental matrix. As you know that even if you move the cameras at any point in the imaging system, a coordinates will coordinates will vary, but still fundamental matrix will remain the same. So, this is one of the interesting property. So, P P prime gives a fundamental matrix F it is unique.

So, the question is that whether we can get the projection matrices P and P’ from a fundamental matrix F. Now, this is a property which we can easily verify and by which we can see that we do not have a unique solution in this case. That means, though projection matrix is a pair of projection matrices P and P‘ give an unique F, but there could be other pairs of projection matrices which will can give me the same fundamental matrix F.

So, you consider the situation that there is a homography matrix in the form of 4x4 homography which means is a linear transformation. So, if I multiply P with a 4x4 nonsingular matrix which is invertible matrix. So, from PH, I can again get P.

$$If\ (P, P') \rightarrow F, then, (PH, P'H) \rightarrow F$$

There is no problem, I can simply multiply with its inverse, but the thing is that if I multiply PH; H with this pair and P’ with H, then I will get the same F. That is a theory.

So, the proof is that. So, if I consider a particular scene point X and consider this stereo system, then the corresponding pair of points will be PX and P‘X. Now, consider a transformation of scene point by this homography 4x4 matrix; that means, by a 4x4 nonsingular matrix a linear transformation. So, you know that X is a 4x1 column vector. So, if I multiply with 4x4, it still remains a 4x1 column vector. So, now, you consider rather I will make H^{-1} instead of H, let me do it H^{-1}. H^{-1} will be also a 4x4 matrix because I mentioned H is an invertible matrix.

So, if I consider another projection matrix PH and multiply with $H^{-1}X$, we will get PX the same points. Similarly, P'H multiply with $H^{-1}X$, we will get P'X which are the same corresponding pairs of point. So, same corresponding pairs of points have these solutions. It could give me a fundamental matrix satisfying no a set of projection matrices in the form of PH P'H and corresponding scene points as $H^{-1}X$. So, this is what this proof is and that is I can summarize once again just to make it a clean display.

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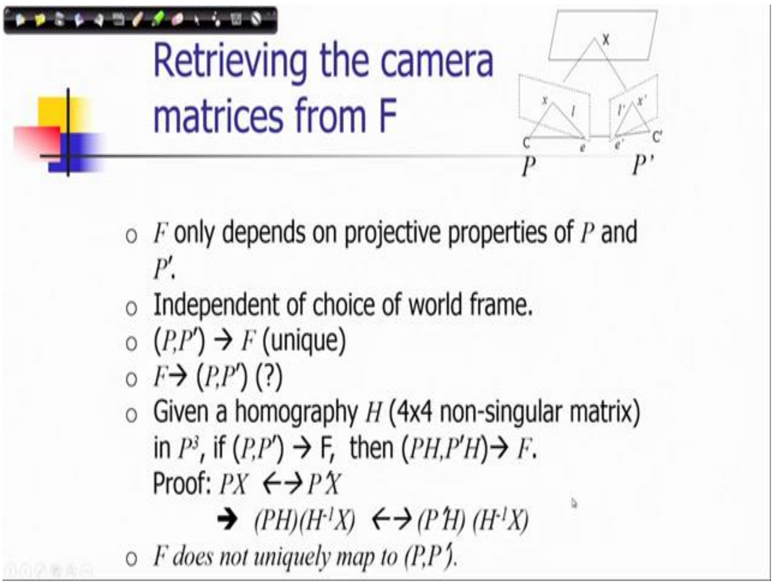

So, this is just we derived; this is the scenario. So, the summary is that F does not uniquely map to P'.

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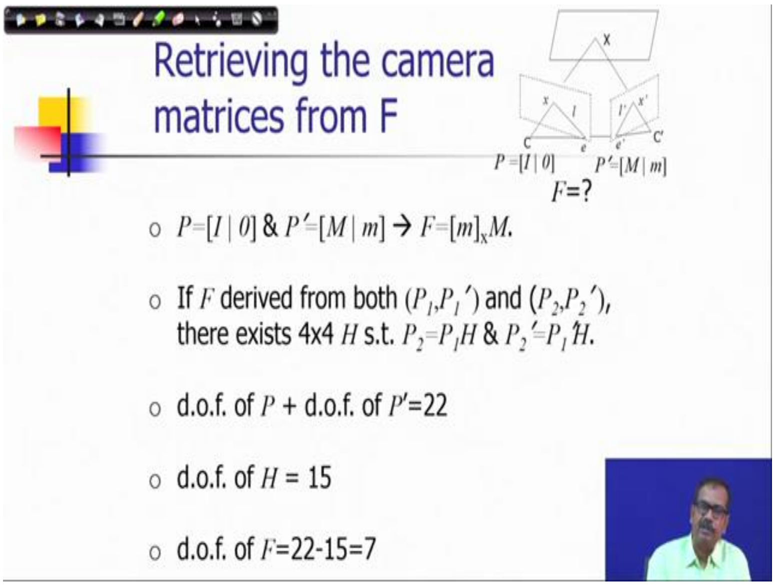

So, once again we continue this discussion. So, let us consider this particular scenario, we discussed earlier that projection matrix is in the form of a canonical from that is for the reference camera which is given by this

$$P = [I|0], P' = [M|m] \rightarrow F = [m]_X M$$

Then we know that given this pairs of projection matrices, its fundamental matrix of the stereo system can be computed using this expression.

Note that this m is a right epipole; that means, this is column vector is the right epipole of this. So, this M provides you also the homography at infinity. So, we note e' cross homography at infinity that gives me a fundamental matrix that is a relationship we are discussed here.

So, we discuss that if we can derive F; if you get F from two camera systems the same F, then there should be a relationships between the projection matrices of these two camera systems. It collaborates with the theory, what we discussed in the previous slide. And we can explain that because of this constraint, we can explain how degree of freedom of F is related to the degree of freedom of projection matrices.

Say we know that number of independent parameters of a projection matrix is 11. Just the scale that it should be scale equivalent that is all. So, there are 12 elements and so, number of unknowns are 11 or number of independent parameter is 11. So, since there are in a stereo system, it involves two projection matrices. So, maximum degree of freedom could be 22, but since they are related by this homography 4x4 homography; once again homograph 4x4 homography matrix is also a projective element. That means, there is it is described upto scale.

So, one of the element could be a scale element. So, there are 15 degree of freedom of H. So, that is the constraint imposed on a fundamental matrix that given that fundamental matrix, projection matrices should be related given any reference projection matrix. So, 22-15 will give you the degree of freedom of F. So, this justify how that degree of freedom of a fundamental matrix is 7.

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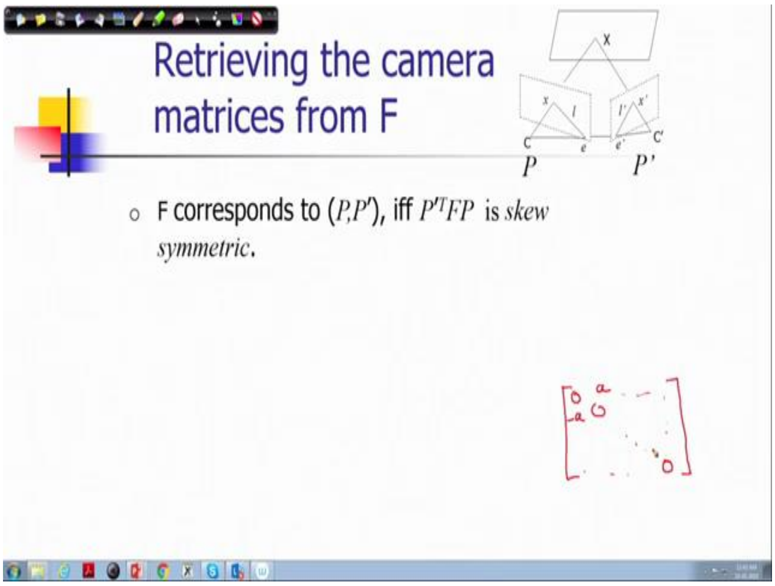

There is another interesting fact that we like to discuss that the fundamental matrix corresponds to a projection matrix P and P' if and only if $P'^T FP$ is a skew symmetric matrix. That means, it check the compatibility. So, if I give you a fundamental matrix and if I give you P and P', immediately I can check with this relationships whether they are compatible or not.

I need not compute F from P and P' to check it. I have to check the scale equivalence there of course, but in this case simply I have to check whether $P'^T FP$ is skew symmetric.

So, what is skew symmetric? Just to remind you, a skew symmetric is a matrix where the transpose of this matrix should be the negative of, this one and the diagonal element would be 0. So, this is one example.

$$A^T = -A$$

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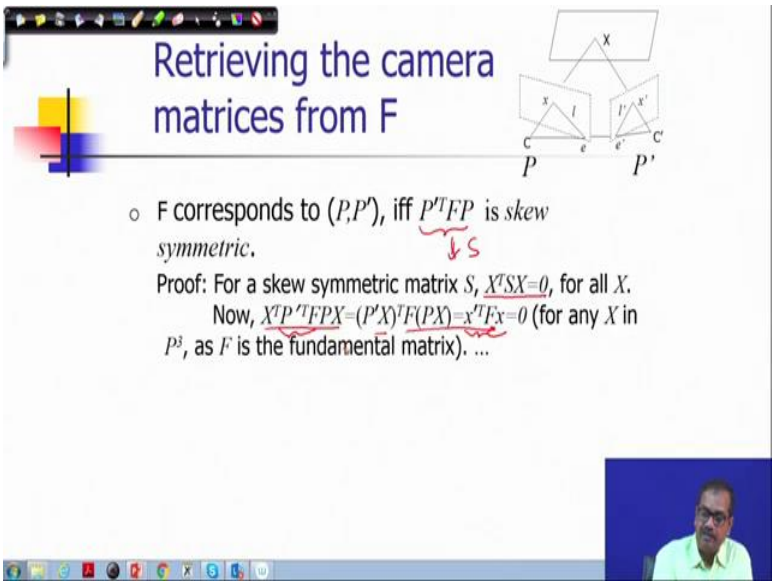

So, just to prove this fact one of the interesting property of skew symmetric matrix that is exploit that if you consider any skew symmetric matrix, then this fact is true; that means

$$X^T SX = 0 \; for \; all \; X$$

You have to choose a proper matrix dimension. So, a skew symmetric matrix has to be a square matrix and when it is true for all X. So, if it is a skew symmetric matrix, it has to be true for all X because of that negation properties that the transpose of this is a negation.

So, now, we will be showing that if I perform this operation,

$$X^T P'^T FPX = (P'X)^T F(PX) = X'^T FX = 0$$

You see that this is true for only the corresponding pair of points, but for any scene point in the three-dimensional phase or in the homogeneous coordinate four-dimensional homogeneous coordinate space, this is true and that is how if this quantity $P'^T FP$; this is a skew symmetric matrix.

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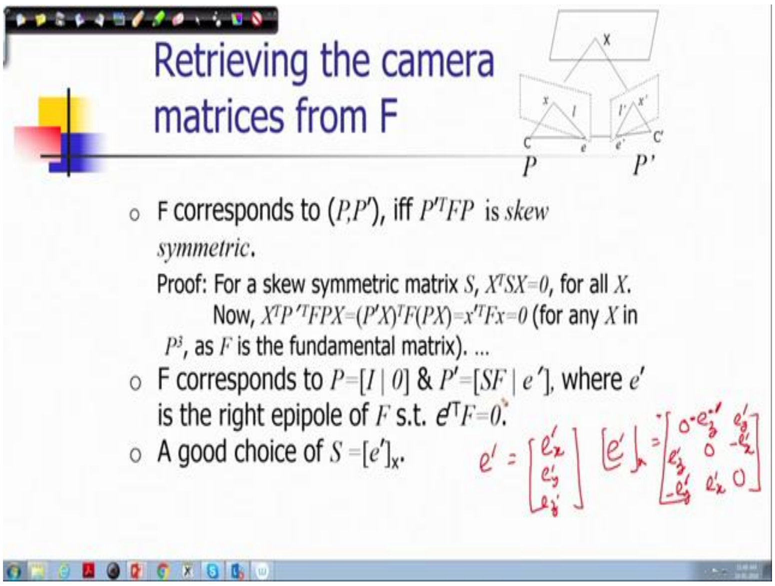

Then another relationship we will be knowing here that F corresponds to P and P', where P' is given in this form. That means, one of them is a skew symmetric matrix; S is a skew symmetric matrix and F is a fundamental matrix.

So, this relation is giving me a particular pair of no projection matrices. It is helping us to form at least a pair of matrices, where which will give me the corresponding F. So, I will not give the details of the proofs, but these results are interesting. So, that we will be writing here. So, one of the good choice of a skew symmetric matrix, we have already discussed that is e prime cross.

We know $[e']_X$ is a skew symmetric matrix and if you remember that if I consider e', if I represent e' as say

$$e' = \begin{matrix} e'_x \\ e'_y \\ e'_z \end{matrix}$$

then $[e']_X$ is given by a skew symmetric matrix in this way.

$$[e']_x = \begin{matrix} 0 & -e'_z & e'_y \\ e'_z & 0 & -e'_x \\ -e'_y & e'_x & 0 \end{matrix}$$

.

So, you see that this is a skew symmetric matrix. So, the solution from F at least we can get one set of one pairs of projection matrices; one of them is in the canonical from [I|0] and the other one would be$[[e']_X F|e']$. So, this is at least a solution you can get from here.

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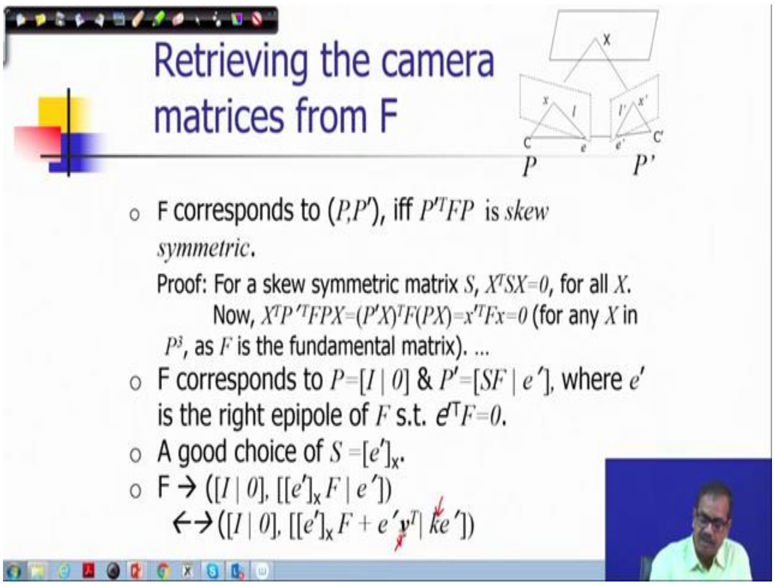

So, this is what I just mentioned that given a fundamental matrix at least you can get a pair of projection matrices in this form. But as we mentioned there should be a family of projection matrices pairs of projection matrices which will give you the same fundamental matrix and this family is shown here also. Once again, the results we will discuss the results, but not the derivation.

So, here we can see that these are the equivalent pair's projection matrices which will give you the same fundamental matrices or this is a family of fundamental matrix. So, what is the definition here. See v is any 3 vector and k is a scalar constant. So, the ke' is the same epipoles that you expect there and e'v^T it is a 3x3 matrix once again you can multiply with.

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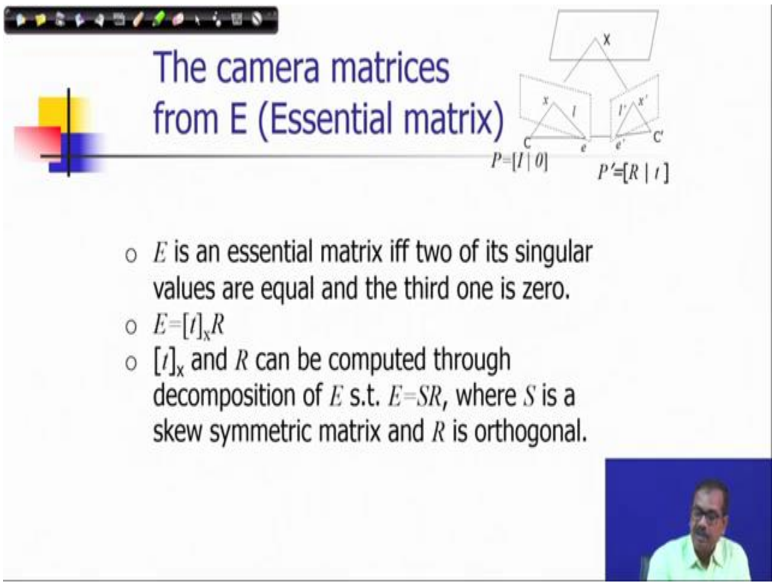

For essential matrix, we can also derive the camera matrices as you can see that essential matrix is given in this form. So, once again this is a $[t \quad]_X$ is a skew symmetric matrix and R is a rotation matrix which is an orthonormal matrix. So, this is a kind of decomposition. So, given an essential matrix, if I get a decomposition of skew symmetric matrix and rotation matrix which is an orthogonal matrix; then I should be able to get at least a pairs of projection matrices given P = [I | 0] and P' = [R | t].

And the other fact we have already mentioned that two of its similar values are equal and the third one is 0. So, the matrix decomposition of E should give me the corresponding pairs of matrices.

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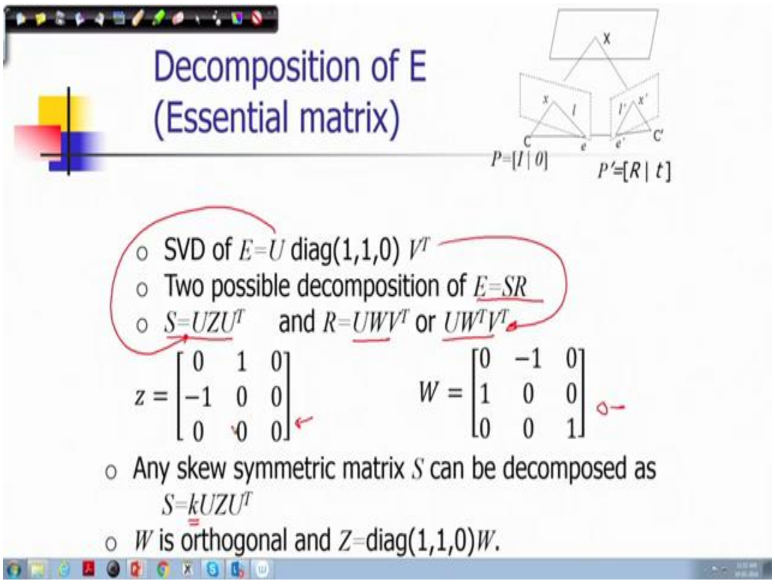

Now, there is a technique which ensures this decomposition. So, we have to perform singular value decomposition of essential matrix in this form, what is given here as you can see and there are two possible decomposition of essential matrix in the form of a skew symmetric matrix and rotational matrix; where you can see that it is a this E = SR and S is given in this from UZU^T;

So, U comes from the definition of the singular value decompositions and R is UWV^T or UW^TV^T. And you can find out the form of z and W.

$$z = \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$W = \begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

So, this is one particular solution. You need to perform the single value decomposition and then, z and W, they are well defined and you can get the corresponding matrices.

So, some more elaborations at any skew symmetric matrix S can be decomposed into this form

$$S = kUZU^T$$

you can show and even if we can multiply with a scale k as I mentioned that always for any fundamental matrix or essential matrix, you can multiply with a scale and you can show that W is an orthogonal, in the orthogonal form and Z=diag(1, 1, 0)W. So, there is a relationship between Z and W, we will see the motivation of why we are explaining this properties.

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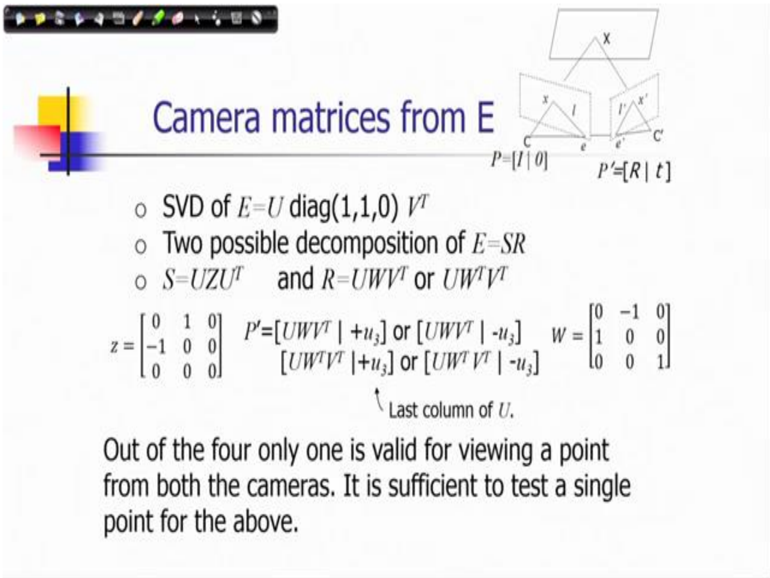

So, by exploiting those properties, these are the possible configurations of the projection matrix of the second camera. So, first cameras projection matrix is already taken as [I | 0] in the canonical form. So, these are the possible configurations as you can see there are four such options are there

$$P' = [UWV^T|u_3] \; or \; [UWV^T|-u_3] \; or \; [UW^TV^T|u_3] \; or \; [UW^TV^T|-u_3]$$

You note the definition of u_3 here which is the last column of the matrix U which is given by this single value decomposition which means u_3 corresponds to the singular value 0 of U.

And but out of this four, only one is valid for viewing a point from both the cameras. We have already discussed that given a camera matrix how you can decide which point is in front of the camera. So, one of this projection matrices will have the same frontal directions, same directions with [I | 0] and you should consider that matrix which satisfies. So, this is one form of solution of projection matrices from essential matrix.

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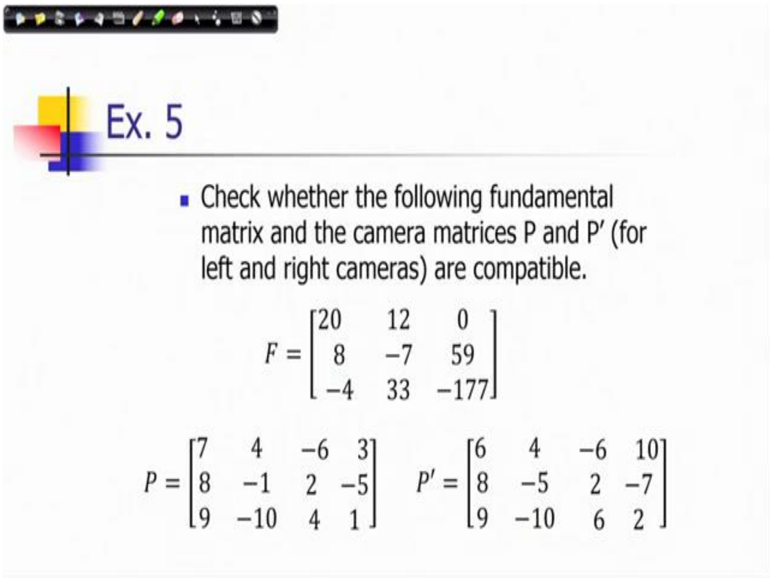

So, let us consider an example here using those properties. Now this problem says that you have been given a fundamental matrix and also the projection matrices P and P' of this stereo vision system. Now you have to check whether they are compatible or not. We discuss this particular property we have to exploit the property that P^T or $P'^T FP$, it should be skew symmetric.

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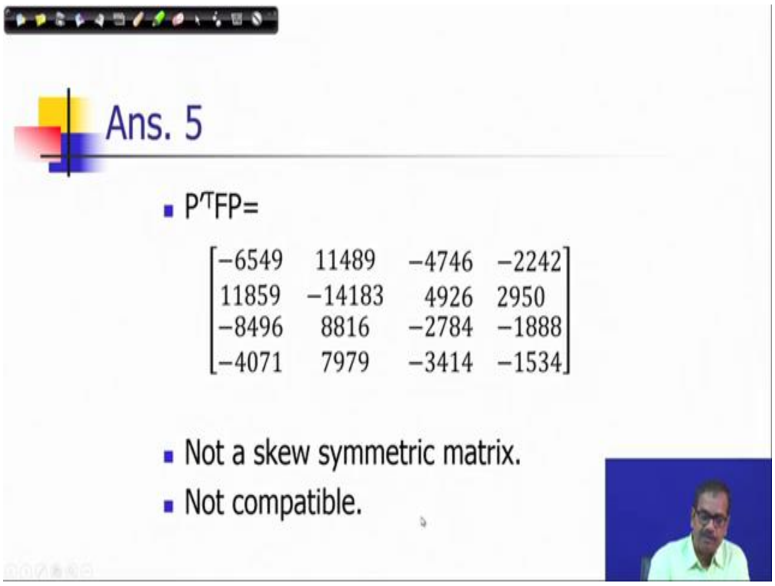

So, we need to compute this particular element, particular entity and if I perform matrix multiplications, I will get this matrix. And you can observe that this is not a skew symmetric matrix. So, which can readily tells us this fundamental matrix is not compatible with this cameras which is not compatible.

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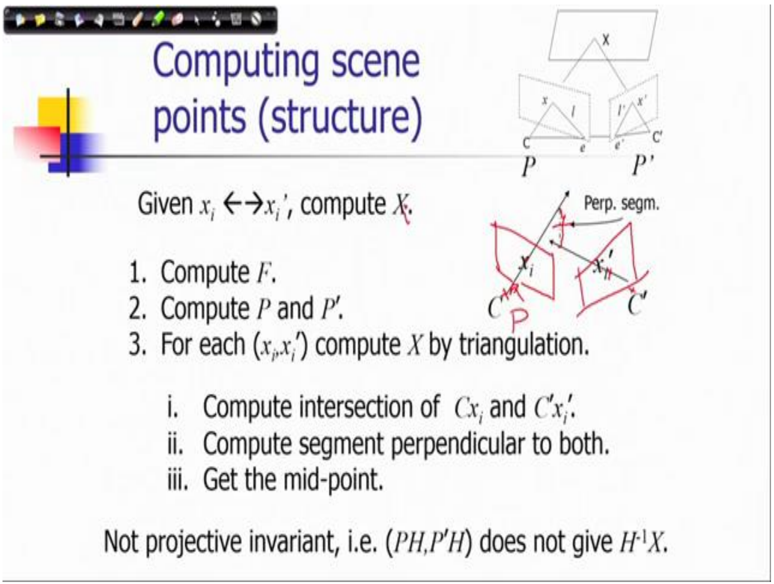

We will now consider another computational problem that given a pairs of corresponding points, a set of pairs of corresponding points; how can you compute the scene points. We sometimes say for the whole and symbol as the structure of an object because if you know the three dimensional coordinates of the object points, we consider we have recovered the structure. So, essentially, we would like to compute the scene point ith scene point I can say also x^i scene point whose corresponding points are given by $x^{i'}$

So, one of the strategy could be that I can easily compute fundamental matrix F. Then, I need to find out P and P'. So, which may not be unique, but at least it gives the possible structure and then, I may later on do post processing to get proper combination of P and P' to get a structure. But right now, let us consider only that whatever P and P' we get, we need to compute the corresponding scene points.

Then we can apply the triangulation. So, I will explain what this method is by which you can get the scene points. So, triangulation method, it says that if I have the scene point x_i. Say you have an image plane corresponding to this camera, whose centre is here C and say this is your image point x_i. So, I can always form given its projection matrix P, I can form the back projection rate. So, I can form the equation of the projection ray, we have discussed earlier.

Similarly, for the other camera I know its camera centre; centre of the camera and its scene point I can form the other projection ray which means I would get the equations of straight lines you need three-dimensional space. So, now, the ideally, they should have intersected to my scene point; but there would be errors due to observation, errors due to computation. So, they may not exactly intersect. So, what I should consider, I should consider the closest point which should have been an intersecting point.

So, consider a perpendicular projection from this line to the other line. So, that is a linear segment and take the middle of that line segment and we will consider that is my scene point that is a solution. So, this is what we will compute intersection of Cx_i and $C'x_i$ front and for that we need to compute the segment perpendicular to both and get the midpoint.

So, we should note that this computation is not projective invariant because as we mentioned that camera matrices could have been (PH, P'H). Of course, if it is (PH, P'H), you would have got $H^{-1}x$. But the thing is that the way you are doing you would not get $H^{-1}x$ if you take also (PH, P'H). So, that compatibility is not there. It would be close to that, but it is not theoretically it will not ensure that you will get $H^{-1}x$.

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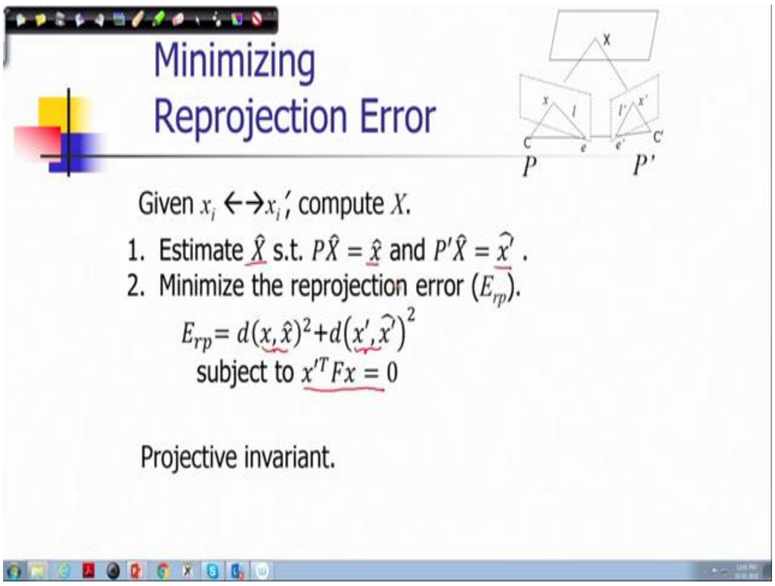

Another method could be that we can estimate the scene point by minimizing certain objective function by considering the projection of those scene points applying the projection matrices. So, in this case you can see that what you can do is start with an initial estimate and then, observe what is your image point. Now, those image points should be close to your observed image point.

Then, this is the image point you obtain from computation and these are the observed one. So, if your estimate is good enough, this error should be very small and they should also satisfy this constraint. So, once again this is a constraint based optimization problem it is a non-linear optimization. There are various non-linear optimization techniques by which you can solve and you can define your estimates and this method is particularly projective invariant. That means if you take (PH, P'H), using this method the solutions will also satisfy $H^{-1}x$ and $H^{-1}x'$.

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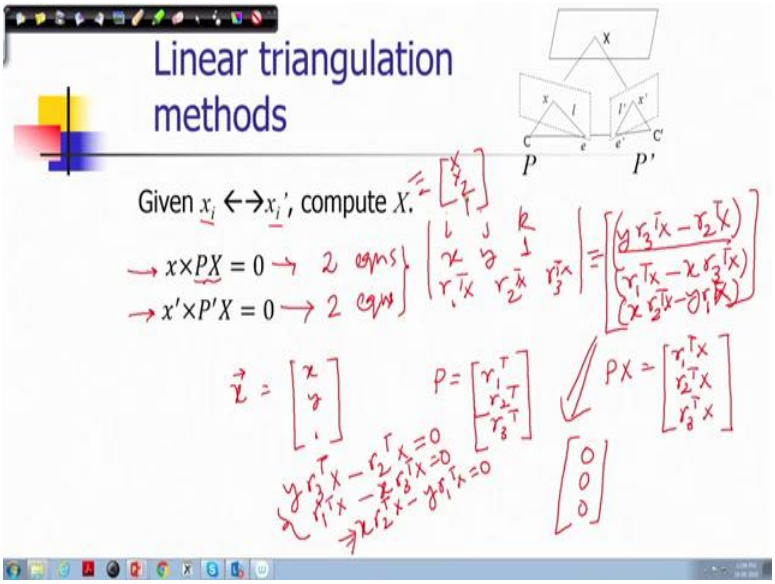

And the linear triangulation methods we have discussed I think this is [FL]. So, in this method we will be using the algebraic manipulations. In the previous case of triangulation, we considered only geometric method of computing the intersections of two lines. But here we will formulate this problem as a solution of a set of linear equations. So, we will form a set of linear equations, as we did in previous cases also you will find out similarity; we will find the similarities of this formulations.

Since we have two pair of points x_i and x_i', we can consider that it forms an equation say Since PX is a same PX gives me the same point, but once again since it is a projective element its it could be scaled version of X, but the direction will remain same as a vector. So, if I take the cross product with X that should be equal to 0.

So, this will give me a set of linear equations to for which satisfies this constraint. Similarly, x'xP'X =0 that would give me another set up equations. So, if I expand this particular form, I can write it in this way say

$$P = \begin{matrix} r_1^T \\ r_2^T \\ r_3^T \end{matrix} \; and \; PX = \begin{matrix} r_1^T X \\ r_2^T X \\ r_3^T X \end{matrix}$$

See you see that this is a 3 vector and similarly, x also a point where we can represent in this form so

$$\vec{x} = \begin{matrix} x \\ y \\ 1 \end{matrix}$$

So, I have to take the cross product of these vectors. So, I will form this equation (x, y, 1), then $r^{1T}x$, $r^{2T}x$ and $r^{3T}x$. This is a cross product and there as you can if I expand these cross product once again. So, you will find that

$$y r_3^T X - r_2^T X = 0, \qquad r_1^T X - X r_3^T X = 0 \; and \; X r_2^T X - y r_1^T X = 0$$

So, we will find actually the third equations can be derived as a linear combination of these two.

You multiply with x and multiply with y and then, if you add them you will get the third equation. So, which means there are only two independent equation. So, it will give you only 2 equations. Similarly, this constraint will give you another two equations. So, given a pair of corresponding point you have 4 equations and how many unknowns you have for x? You have only 3 unknowns because you know these are the world coordinate points x, y, z and there is the scale dimension. So, there are 4 equations and 3 unknowns. So, once again you can apply the least square error estimate technique that we did.

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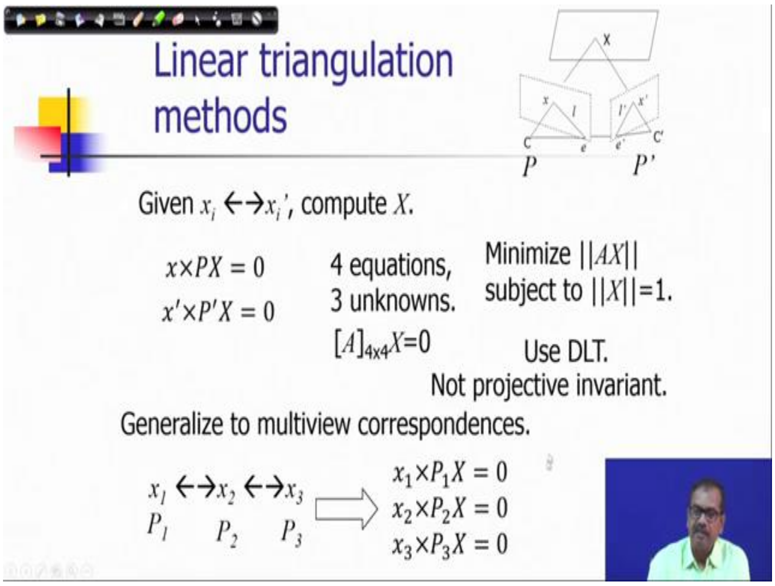

So, just to summarize no this particular computation, we can consider that there would be 4 equations, 3 unknowns as we mentioned and this can be expressed in this form set of linear equations. So, once again the same it is a set of homogeneous equations. So, we have to minimize the norm AX subject to norm X to be equal to 1 that is the constraint we will impose.

And you can apply this direct linear transform that we discussed earlier also for estimation of projective transformation matrix or homography matrix and this is not a projective invariant. We can generalize this particular construct that is the power of this technique. Because, if you have 3-point correspondence; that means, you have a three-camera system say P_1 P_2 P_3. So, each one will give me the similar form of equations. So, each one means each one will give me 2 equations. So, there are 6 unknowns; sorry 6 equations, but there are again 3 unknowns. So, you can solve using the same technique.

So, I think with this let me stop here we will continue this discussion of structure recovery in our next lecture.

Thank you very much.

Keywords: Camera matrix, Projection matrix, fundamental matrix, triangulation.