Stochastic Structural Dynamics
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Lecture No. # 08

Random Processes-3

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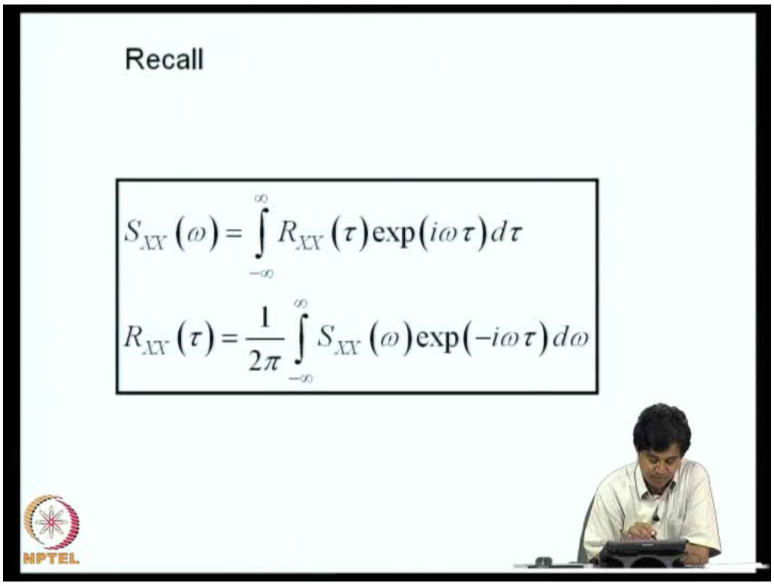

In the previous lecture, we considered frequency domain description of random process. Especially, the stationary random processes, as we saw where (( )) for a frequency domain representation in terms of what is known as power spectral density function? The power spectral density function and auto covariance function, we showed that they are related through these relations. Actually, they form a Fourier transform pair, we demonstrated various properties namely, the power spectral density function is symmetric is even function in omega, it is non-negative and area under power spectral density function is the variance of the process. And therefore, variance of the process we gave an additional meaning that it is also the average total, average power in the signal.

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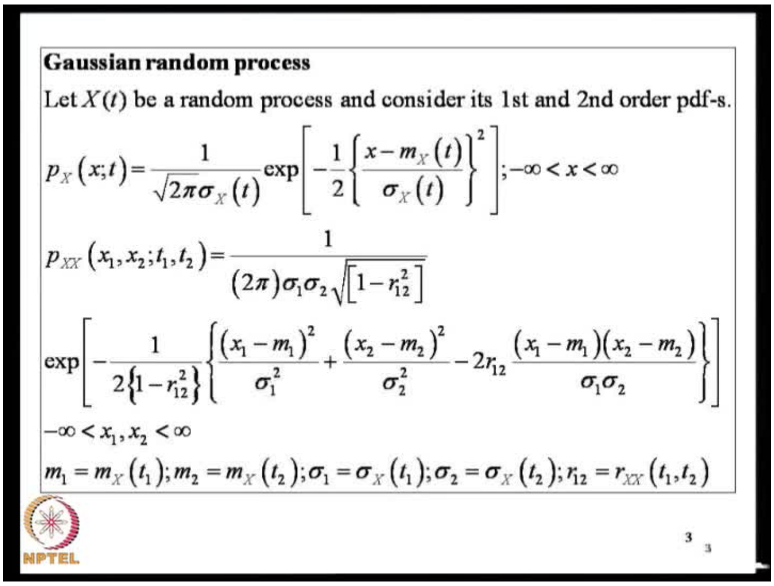

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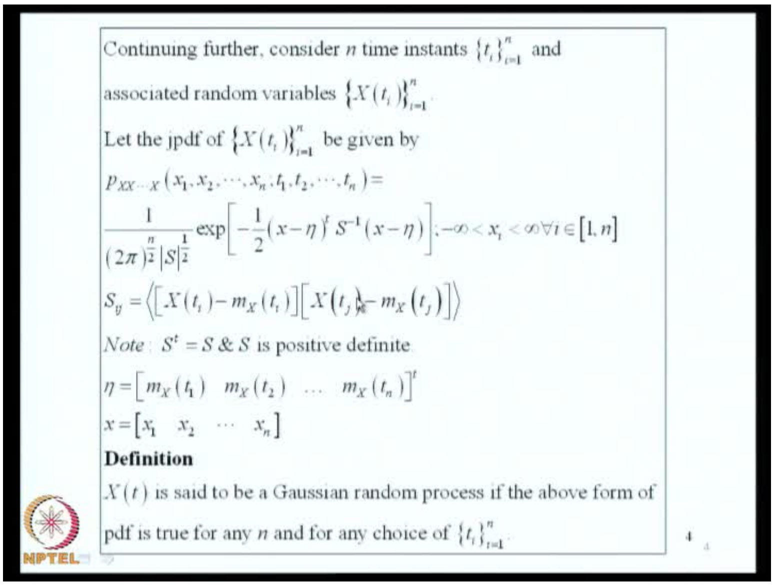

We will continue this discussion and we will return to these descriptions in context of certain random processes. In one of the pervious lectures, we have seen the definition of a Gaussian random process and if you recall, if we consider X of t to be a random process and if there first and second order density functions have these Gaussian forms. This is a probability density of a Gaussian random variable. This is the probability density function of a pair of a Gaussian random variables and if we now consider n time instants and consider the n random variables X of t 1, X of t 2, X of t n and look at the nth order joint probability density function, this is the form of the nth order probability density function of vector of a Gaussian random variables.

If we say that X of t is the Gaussian, if the above form of probability density function is true for any n and for any choice of t 1, t 2, t 3 and t n, that mean you select any t 1, t 2, t 3, t n, you will get n random variables and write down the nth order probability density function, for those random variables and if that probability density function has this form for your any choice of the t 1, t 2, t 3, t n and any choice of n, then we say that X of t is a Gaussian random process.

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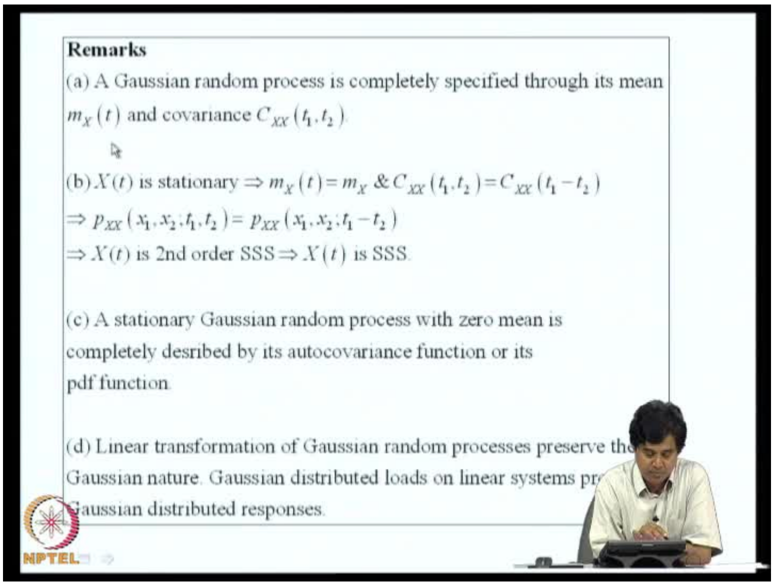

A Gaussian random process therefore, would have basically a mean vector and a matrix of covariance. Thus a Gaussian random process is completely specified through its mean and auto covariance function t 1 to t 2. Now, if X of t is stationary, then we say that mean is constant and the covariance the function of time difference. So, the probability density function is joint probability function will be given by x 1, x 2 is t 1, t 2 is function of t 1 minus t 2. So, X of t is second order strong sense stationary, would mean automatically here it is strong sense stationary in all order. Actually, X of t is weak sense stationary implies it is strong sense stationary second order. And the second order strong sense stationary implies strong sense stationarity in all order. This is because of the intensitic structure of the joint density function, because the joint density function basically involves only mean and auto covariance functions.

So, thus we can now, say that a stationary Gaussian random process with zero mean is completely described by its auto covariance function or its PSD function, because if you know auto covariance, we can always find out this. This is PSD function, this not PDF PSD function, if you know the autocovariance function is Fourier transform is the PSD function. Therefore, either this or this would completely specified a Gaussian random process.

So, this makes the auto covariance and power spectral density function as the most important modeling tools, if you are dealing with Gaussian random process, because they offer the complete description of the random process. While on Gaussian random processes, we have to state this probability namely, linear transformation of Gaussian random processes, preserve the Gaussian nature. This linear transformation could be algebraic are through a differential operator or integral operator that means, if you have Gaussian inputs to linear systems, the output will also be Gaussian. This is a very important property and thus in the context of linear random vibration problems, Gaussian random processes play a very crucial.

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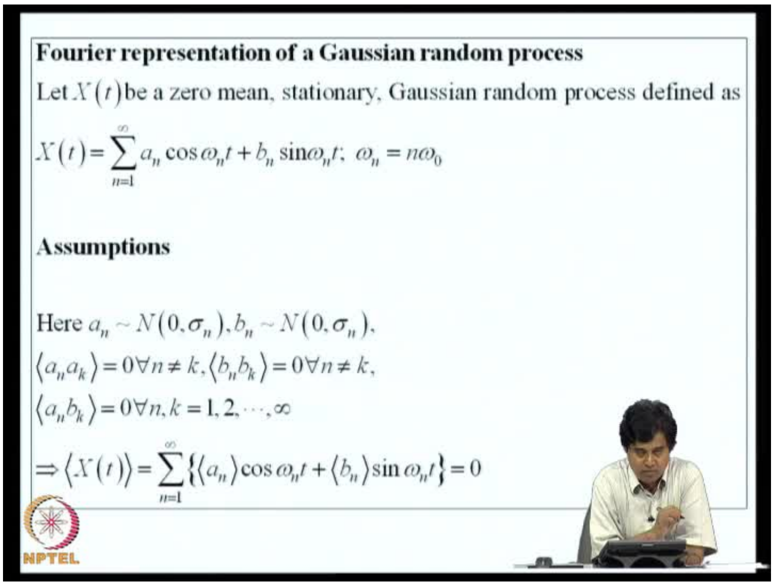

Now, let us consider the problem of the Fourier representation of a Gaussian random process. Now, I define X of t to be a zero mean, stationary Gaussian random process through this relation, it is not a representation for given Gaussian random process, but a construction of a Gaussian random process, in terms of sin and cosin function, here a n and b n are random variables, which are specified; a n I take it to be normally distributed is zero mean, a standard duration sigma n and b n to be normal distributed is zero mean; the standard deviation sigma n. I assume that a n and a k are independent so or b n and b k that means, the expected value of b n b k 0 for a not equal to k and similarly the family of a n random variables and b n random variables that taken to be independent that mean all this random variables are independent and actually identically distributed as well because there normally distributed.

Again, I want emphasize that I am constructing a Gaussian random process, I am not representing a given the Gaussian random process. In terms of this right hand side, I am selecting a n and b n and arriving a description of x of t. Now, what would be the mean of X of t, mean of X of t would be summation of mean of a n cos omega n t plus mean of b n sin omega n t; since mean of a n and b n are 0 for all n mean of X of t would also be 0.

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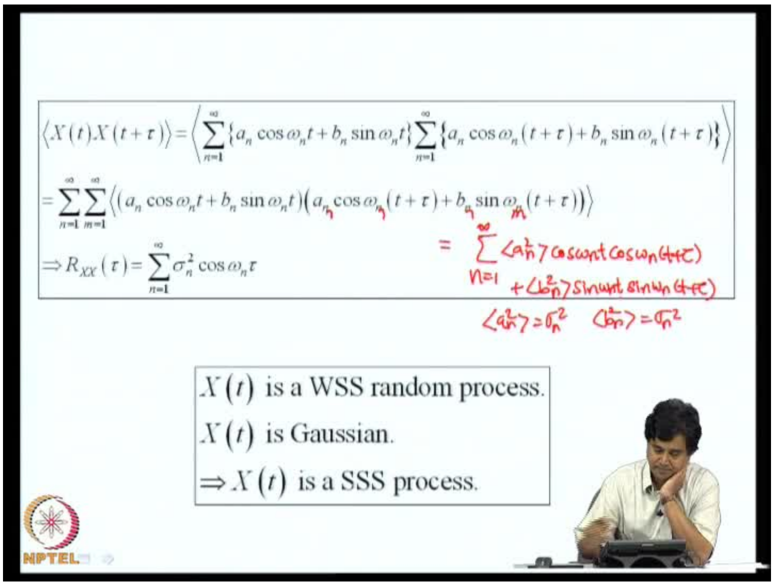

Now, what happens to the auto covariance function? So, let us consider expected value of X of t into X of t plus tau. This is X of t this is X of t plus tau. So, there now this becomes a double summation. So, if I write it as double summation this is the expected value of this term here. Now, if you operate the expectation operator this should be a m this m, this m, this is m. So, a n and a m are independent a n and b m are independent, b n a m are independent and b n b m are independent that means, this double summation will now collapse to a single summation. So, this would be n equal to 1 to infinity expected value of a n square cos omega nt cos omega nt plus tau plus b n square sin omega nt sin omega nt plus tau; this double summation collapse to this single summation. Now, we know that expected value of a n square is sigma n square and b n square is also sigma n square. So, I can pull that out and I get cos omega n t into cos omega n t plus tau plus sin omega n t sin omega 1 t plus tau sigma m square cos omega n tau this cos of a b minus a. So, this is this.

From this, conclude the covariance is now a function of the time shift and not function of t, it is the independent of the t. Therefore, it is a wide sense stationary random process. Since we are performing linear transformation of Gaussian random variables it would follow that X of t is Gaussian. Now, X of t is Gaussian and it is a wide random process. So, it will be automatically means X of t is a strong sense stationary random process.

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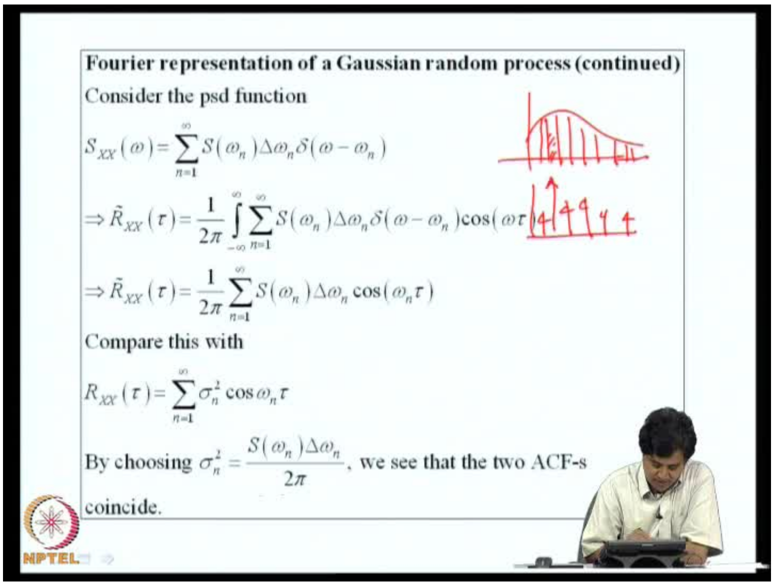

Now, will continue this, now let us consider a power spectral density function, something like this. So, I discretize this into the frequency access into this intervals and I will consider, this area to be represented as a direct delta function. So, this I will write it as series of direct delta functions and I write S XX omega in this form, where s of omega n into delta omega n is an area under these rectangles.

Now, what do will be a its Fourier transform, what will be the auto covariance associated with that. Now, you substitute that into the expression for auto covariance function and you can show that. This auto covariance function will be of this form and if you now compare this R XX tilde of tau with the auto covariance that we derived just now, you will see that these two are equivalent provided it sigma n square is defined as this; that means the two random processes are equivalent in terms of auto covariance function.

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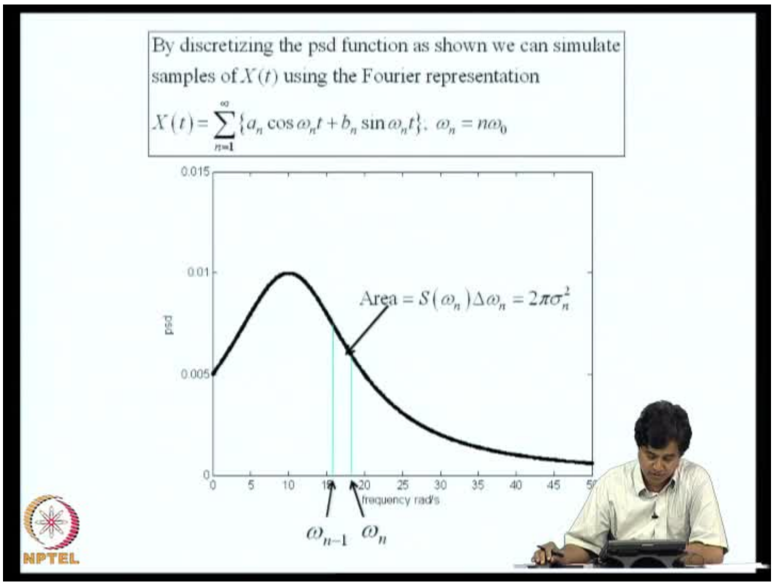

So, what is the application of this description? So, suppose if you are given a power spectral density function, as I was mentioning, I will divide this into segments and this area is S of omega n and delta omega n. This will be 2 pi sigma n square. So, I will construct now, a Gaussian random process, as shown here and I will define now variance of a n and b n to be equal to this and derived from this. This is the nth interval. So, if I generate X of t using this description, we will see that the power spectral density of X of t would converge to the discretizing version of power spectral density of this and as a number of divisions become large, we can expect that the power spectral density X of t and this converge.

This is a useful tool for simulating samples of Gaussian random processes on computing. So, what we have to do this, given power spectral density function. we have to discretizing and find out these areas and defined this random variables a n and b n as having zero mean and variance is which vary as per the shape of the this power spectral density function. And on a computer we can easily simulate samples of Gaussian random numbers, using that I can now simulate samples of X of t; whose power spectral density in some sense will be equal to this.

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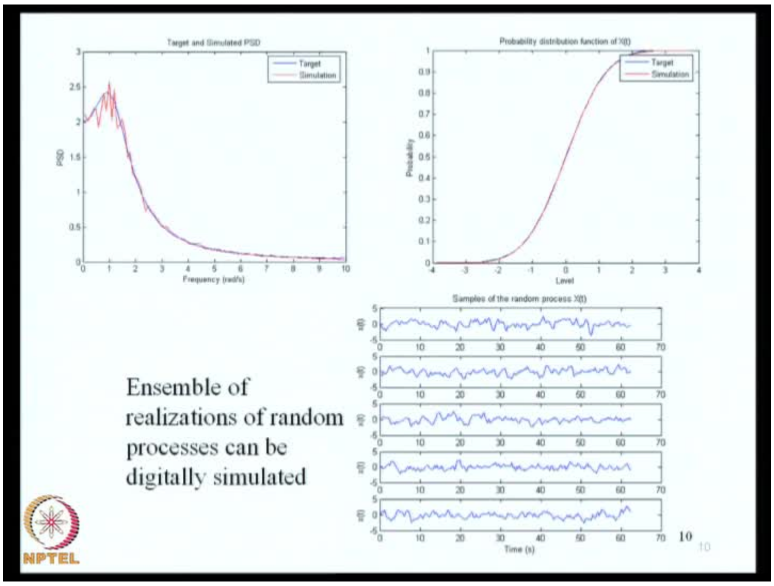

So, this will be a useful tool for what are known as Monte Carlo simulation studies, which we will consider later in this course. So, this is an illustration of a calculation that will actually perform, this blue line that you see here is a target power spectral density functions. Based on this, we simulate samples as shown here, this is a first sample and second sample and so on so forth. In from these samples, we have estimated the probability distribution function and that is shown here. There are two lines here one is red another is blue; blue is a target probability distribution function, and red is what we have actually estimated from this simulated samples and as you can see they coincide with each other very nicely.

Similarly, from simulated samples of X of t, we have estimated the power spectral density function and that estimate sample estimate is shown here and you can see that the red and blue lines follow broadly similar trends. So, this issue we will consider later as I was mentioning, when we considered Monte Carlo simulation methods for analyzing randomly vibrating systems.

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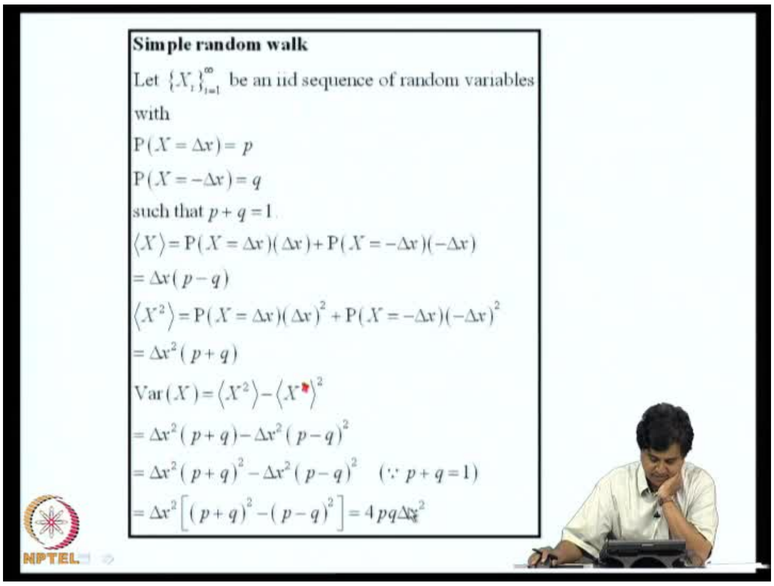

We will now consider, another random process known as simple random walk, we construct this random process as follows: we start with a sequence of identical and independent random variables x 1, x 2, x 3, x infinity, these random variables have a common probability distribution function and they assume only two states probability of X equal to delta x is p and X equal to minus delta x q with the condition that p plus q is 1. So, the sample based consists of delta x and minus delta x.

What would be the expected value of X. So, this the baronial random, what will be the expected value? Expected value is probability of X equal to delta x into delta x plus probability of this into minus delta x. And this is delta x into p minus q. What will be the mean square value delta x square into probability of X being equal to delta x plus minus delta x square multiplied by the probability of X equal to minus delta x. So that would became delta x square into p plus q.

What is variance of X? Variance of X is mean square value minus the square of the mean. This is square of the mean so, this is if you simplify this delta x square p plus q minus delta x square p minus q whole square and if you rearrange this terms slightly, I will write p plus q is p plus q whole square. Since p plus q is 1, I can do this that would enable me to use this relation is a square minus b square and if I use that I get 4 p q delta x square, this a variance.

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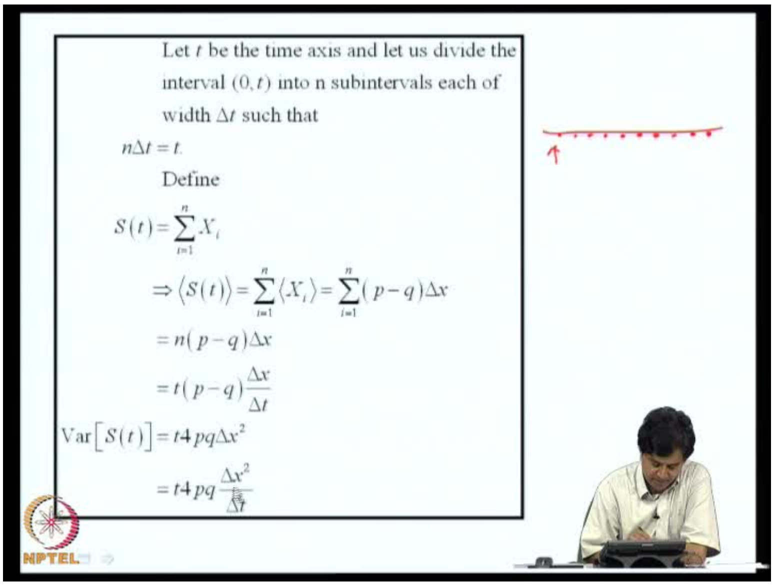

Now, I will now introduce a time axis and divide the time axis the interval 0 to t into n sub intervals of each width delta t. So that n delta t is t, I now define a random process S of t as summation of i equal to n X i.

So, what it means, we have time axis divided into time intervals like this. So, at every time instant, I will say toss a coin; with those two possibilities and if I get head, I move forward, I take a step forward if I get tail, I move backwards. So, S of t is my position after n such trails. So, that means I am performing a random walk on this line, at any given time will toss a coin, if I get head, I am take a step forward, if I get tail I take a step backwards. So, if I keep on doing it as after n such tails, where I am my position that is given by S of t. So, that is given by this.

So, what is mean of S of t mean of S of t is summation of mean of x i and that is i equal to 1 p minus q delta x and this is n into p minus q into delta x. What is n? How did I define n? n is t by delta t here, I will write for n t by delta t and therefore, I can write it as t into p minus q into delta x by delta t. What will be the variance? Variance will be t into 4 p q delta x square, I will write this as t into 4 p q delta x square by delta t. This is n into 4 p q delta x square. This n is t by delta t delta x square by delta t.

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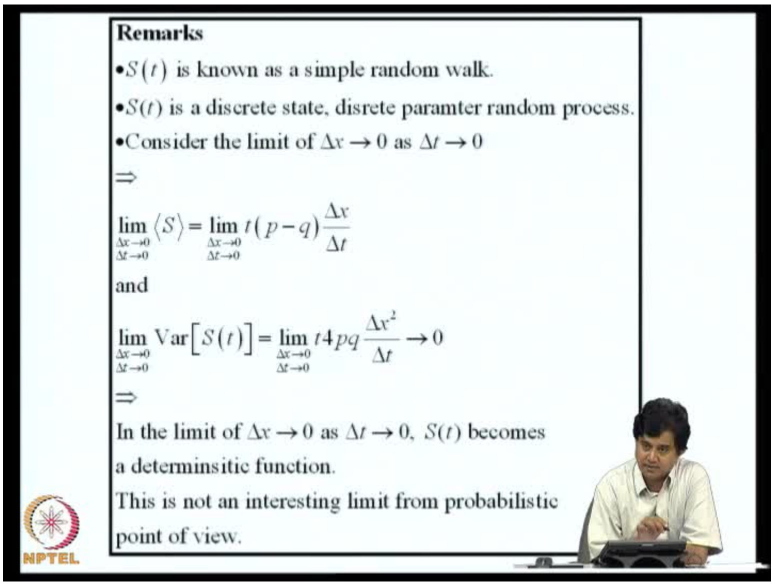

This process S of t is known as a simple random walk, see time axis has been discretizing and the states are in terms of steps. So, it is a discrete state, discrete parameter random process the states will be multiples of delta x. Therefore, it is a discrete state, discrete parameter random process. Now, we will consider, what happens as delta x going to 0 and delta t going to 0. So, what happens to the mean? As delta x going to 0 and delta t going to 0, the mean of this random process will be limit of delta x going to 0 delta t going to 0 this quantity. This can be a meaningful quantity; it will not be an infinity are 0, it will some meaningful quantity, but what happens to the variances? As delta x going to 0 and delta t going to 0 variances of S of t happens going to 0, because as delta x going to 0 at the rate of delta t going to 0. So, there is this delta x by delta t into delta x so this will go to 0.

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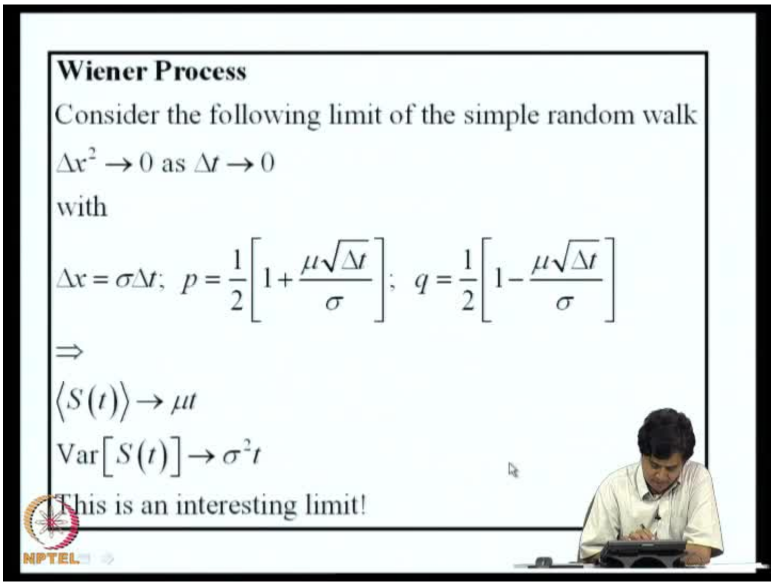

That means, if you consider the limit of the delta x is going to 0 and delta t is going to 0 the resulting random process will have zero variance, or in other words, it will be a deterministic function from the point of view of probabilistic modeling. This is not an interesting limit, because what was actually a discrete state, discrete parameter random process. Now, becomes deterministic function. On the other hand, if we use an alternate, alternative limiting definition what we will do is - we will take delta x square goes to 0 as delta t goes to 0 and we will may assume certain relation delta x is sigma delta t the p is half of this were mu and sigma are properties, which will have to clarify and q is this.

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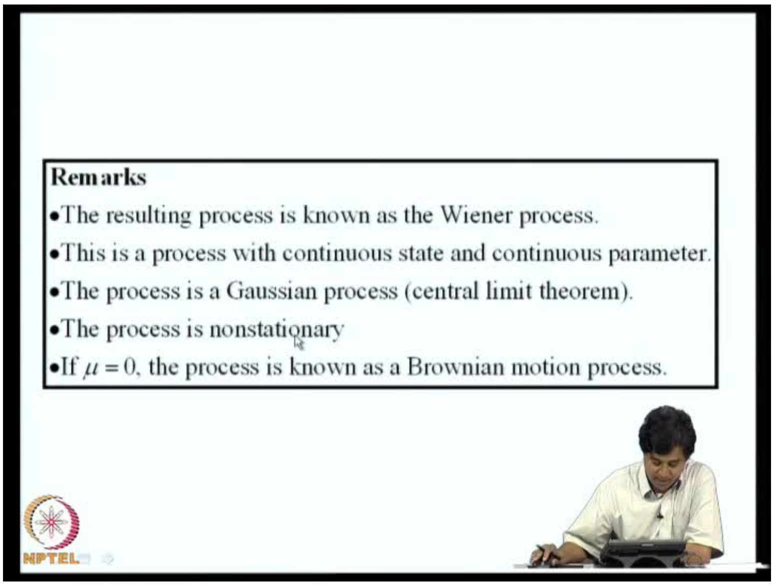

In this case, you can show that the mean of S of t goes to the quantity mu t and variance goes to another quantity sigma square t. Thus, this becomes an interesting limit in the sense a random walk under this limiting situation delta x square goes to 0 as delta t goes to 0, becomes a meaningful random process. This random process as continues time and continues states. This process is known as wiener process. This is a process with continuous state and continuous parameter, as I mentioned just now.

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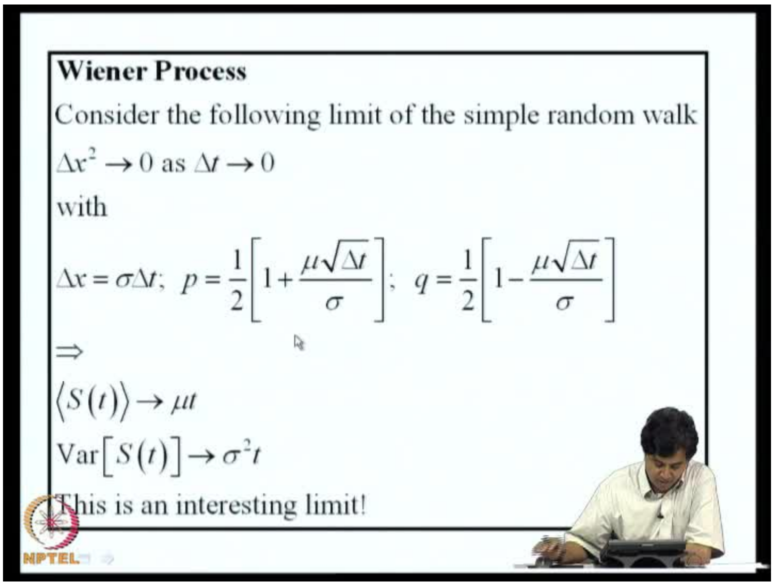

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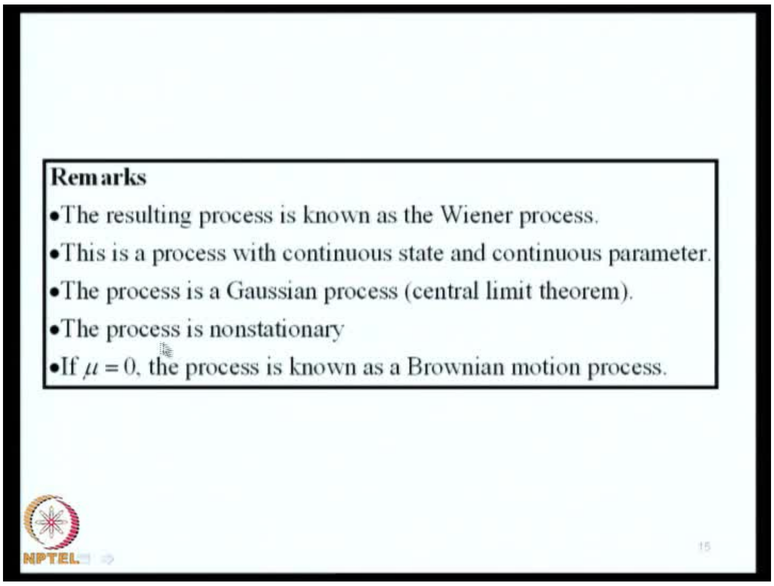

Now, this process we can show the it is a Gaussian process, because we can invoke this central limit theorem, we are adding random variables S of t is nothing but summation of this x is so, as the number of such a random variables be added becomes large. We have seen that we can apply central limit theorem, in the process becomes Gaussian in nature the process is non-stationary why? Because the mean is a function of time and variance is also a function of time, the process is non-stationary, because the mean and variance are time dependent. So, the wiener processes Gaussian non-stationary process, if the parameter mu is 0, we say that this wiener process is as a Brownian motion process. The underlying limits please bare in mind that it is delta x square and delta t are comparable. So, the calculus of Brownian motion process, we will have to taking into account this special feature.

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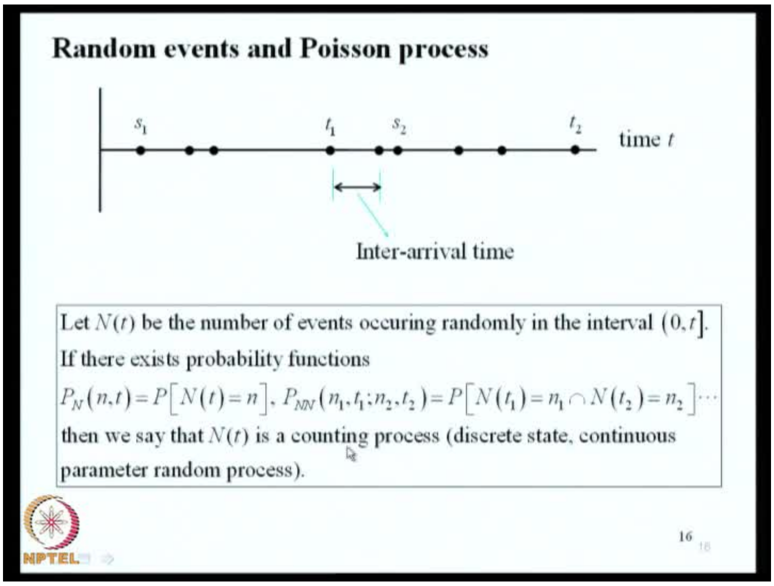

We will return to that question some time later. Now, we will move on to another random process name namely, Poisson process. So, to develop this description of a Poisson random process, we consider a time continuum along which certain acceleration we are taking place, these events could be for example, stress at a point exceeds some limiting value in a structure. So, if I define N of t to be the number of events occurring randomly in the interval 0 to t and if there exists probability density functions P N of n comma t is probability of N of t equal to n and if you consider two time instant t 1 and t 2 probability of p of NN n 1, t 1, etcetera is a probability of n of t 1 equal to n 1 intersection n of t 2 equal to n 2, etcetera. Then we say that n of t is a counting process.

Here, the states are discrete, but time is continuous, the parameter is continuous. The states are discrete because we are counting the states are in terms of integers, where this time is continuous. So, such processes are known as counting processes.

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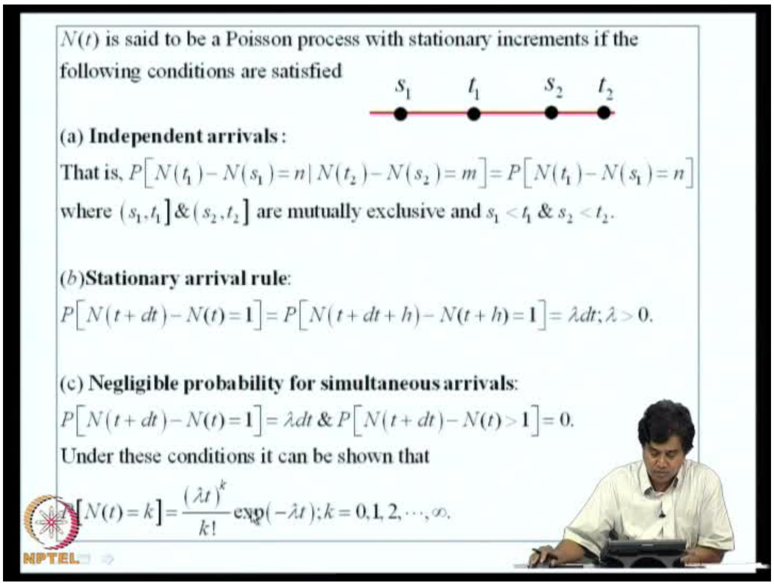

Now, a counting process N of t is said to be a Poisson process with stationary increments, if the following conditions are satisfied, what are these conditions? The first condition says that the arrival should be independent. What it means is the probability that N of t 1 minus N of S 1 equal to n given that n of t 2 minus n of S 2 m is probability of N of t 1 minus N of S 1 equal to n that means these two events N of t 1 N of S 1 equal to n N of t 2 minus N of S 2 m are independent. So, here S 1, t 1, S 2, t 2 are shown here these are order in this manner S 1 is t 1 is greater than S 1 and t 2 is greater than S 2.

Next condition, we place is what is known as stationary arrival rule, the probability of N of t plus delta t minus N of t equal to 1 is same as probability of t plus delta t plus h minus N of t plus h equal to 1 is lambda delta lambda dt that means, by shifting the time by a factor h. This probabilities would not change. So, this is stationary arrival rule. And we do not expect simultaneous arrivals in a sense negligible probability for simultaneous arrivals that means, probability of in the time interval t plus delta t to t to t plus delta t, if there is one arrival. This is a probability of that and more than one arrival is 0, the short time interval for example, in a football match, we rule out the possibility of more than one goal getting scored or stress at a point in a structure exciding the threshold value more than once within a short interval is 0 that means is a rare event it would not happen simultaneous arrival are not possible, using these three conditions, we can derive the probability distribution function of N of t equal to k, we can show that this as this form of a Poisson random variable.

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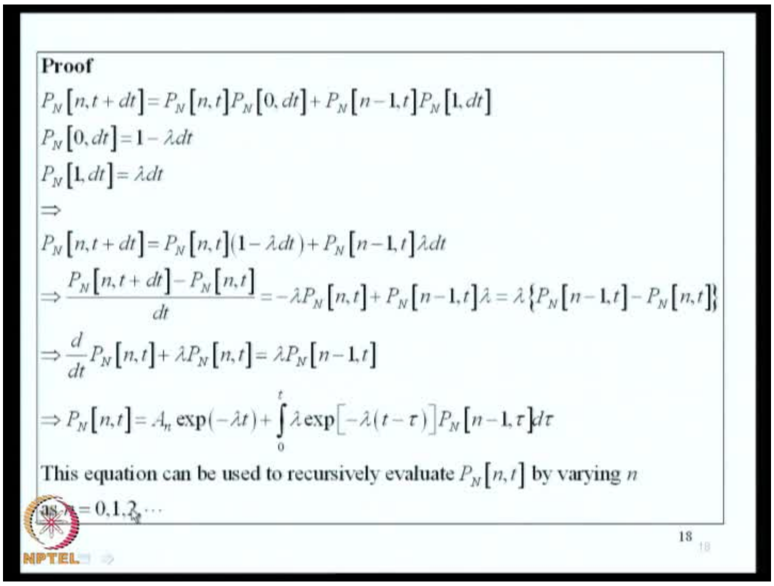

How do we do that, we consider P of N in the t plus d t that means, n arrivals in time 0 to t plus delta t is same as n arrivals in 0 to t with no arrivals in t to t plus delta t plus n minus arrivals in 0 to t plus one arrival between t to t plus delta.

We know, now P of N 0 of dt 1 minus lambda dt and one arrival within short time is lambda dt, where lambda is a rate of arrival. Now, based on this, we can rearrange these terms and for this right hand side, I will use some of this and by rearranging this, we get a differential equation for P N n of n, t d by dt plus lambda this equal to this.

We should notice here that n is also changing there is a kind of differential equation and in the differential equation as well. So, we can solve this equation, the differential equation can be solved. This is a complimentary function, this a particular integral and we can based on this, we can use this solution recursively and evaluate PN of n of t by varying n from 0, 1 and 2 etcetera.

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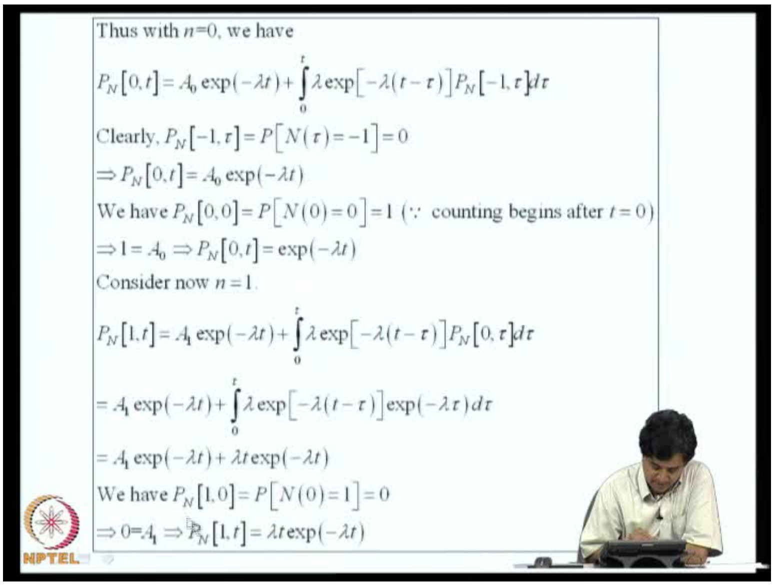

How do we do that we started with n equal to 0, we have P N of 0, t is this, this is this, and I have here in the integrand P N of minus 1, tau that means at probability of n of tau is minus 1 is 0 and P N of 0, t .Thus, I get as a not explosion lambda t. We clearly have P N of 0, 0 at to be 1 because we were beginning counting after t equal to 0. So, a t equal to 0, itself there is no count. So, based on it I get A naught equal to this and P N of 0, t equal to this.

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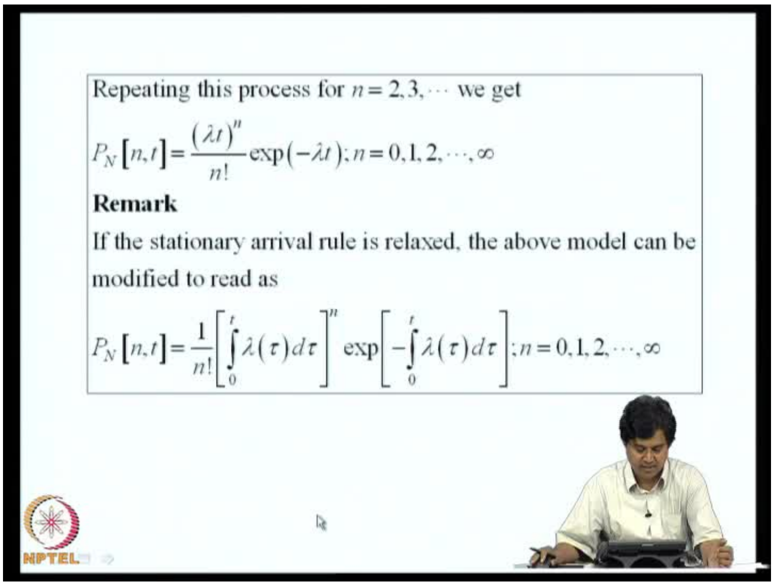

Now, you can now consider n equal to 1. This goes to the right hand side here, and we can integrate now and if you follow this, there will be a constant of integration and when you solve this differential equation here, and that can again be evaluated in a recursive manner and we can show that the requisite probability mass function has a Poisson structure.

If the stationary arrival rule is relaxed that means, the lambda is a function of time also then we can show that this distribution, this mass function will be given in this form, where this lambda t now, it replaced by an integral lambda tau d tau, this is this, will be valid for non-stationary problems.

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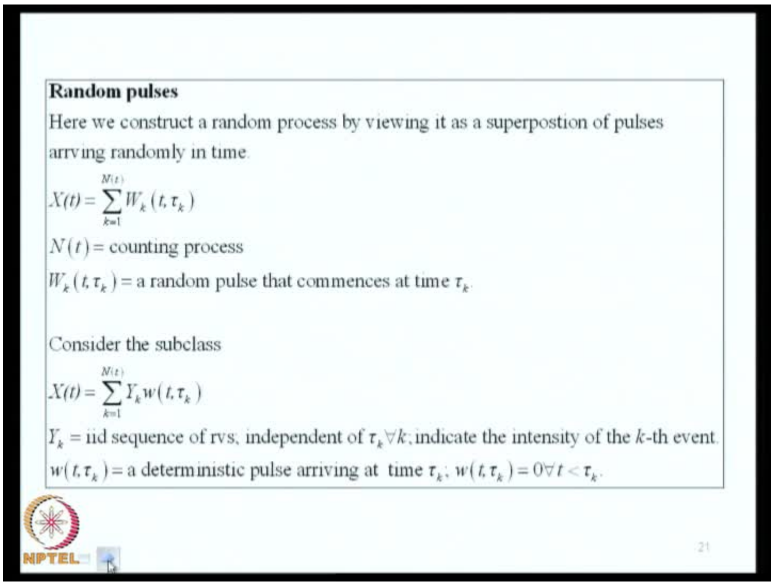

In many situations, we encounter random process, what are known is random pulses? Here we construct a random process by viewing it as a super position of pulses arriving randomly in time. So, we define X of t as k equal to 1 N of t W k t, tau k, where N of t is a counting process and this W, you can imagine in that it is a random pulse that commences at time tau k.

Now, we can make like this slightly simpler and we will consider a sub class, where we write this pulse as Y k into deterministic function W. This is a Y k is a iid sequence of random variables and they are independent of time with which such things arrive and it indicates the intensity of k th event and this w is deterministic pulse arriving at tau k, it is 0 before t equal to tau k.

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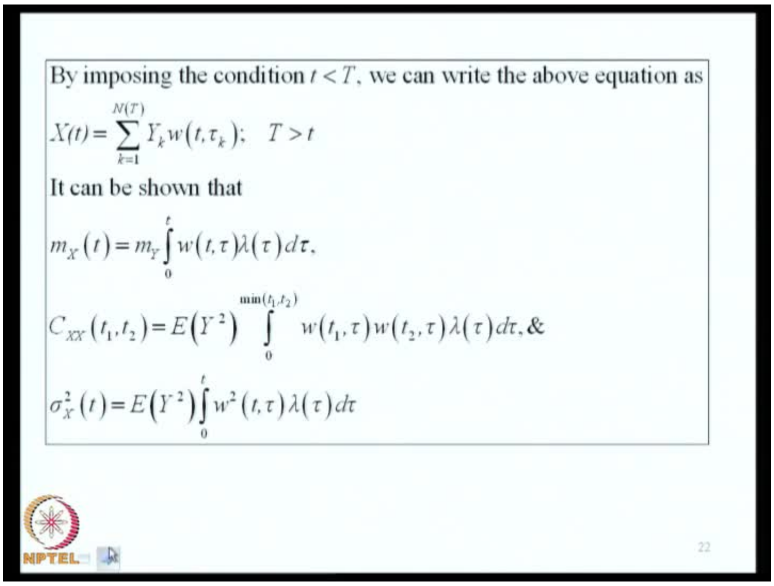

Later on, I will give physical example, for this based on modeling earthquake loads but right now, we will continue the mathematical description. We can right, therefore, X of t in terms of the pulse under deterministic pulse function shape of this pulse, in this form and I will show this later that mean of X and covariance of X will have this form and this will be a useful modeling tool for modeling non stationary random processes. So, you will see more of this later.

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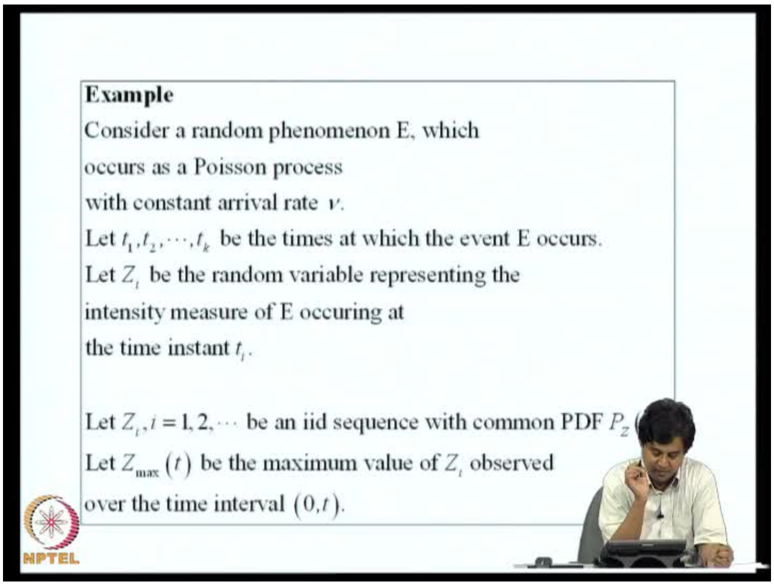

We will now consider an example, will consider a phenomenon E which occurs as a Poisson process with a constant arrival rate nu. This could be for example, occurrence of an earthquake crossing magnitude 5 in a given region or occurrence of (( )) velocity crossing sudden threshold value, it can be any such event.

Now, let t 1, t 2, t k be the times at which the events E occurs, let Z i be the random variable representing the intensity measure of E occurring at the time instant t i, how strong is that event that is captured in the random variable Z i.

Now, we will take this Z i to be an iid sequence with a common probability distribution function, P Z of Z, I am now basically interested in finding out the maximum of this Z i over the time interval 0 to t.

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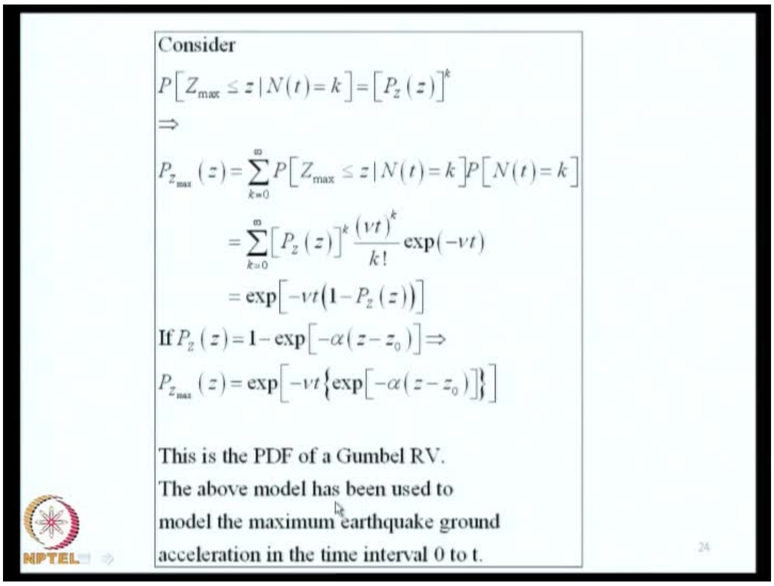

So, now, let us consider, what is the probability of Z max begin less or equal to Z given that there are k even Z are occur. So, we are taking maximum over k random variables which are iid sequences. So, we already shown that this is P Z of Z to power of k, when we discuss extreme of sequence of random variable, we have done this exercise.

Now, the unconditional probability that probability of Z max less or equal to Z is obtained by summing this are k from 0 to infinity and I get this relation. So, this itself is in terms of the probability distribution of Z can be written in this form. This is Poisson of counting process taken to the Poisson, therefore, this is an expression. Therefore, now I get this maximum as exponential of this where in the exponent I have a density function.

Now, if this density function is exponentially distributed, we get probability distributions for the maximum in terms of exponentials of exponential. This is the form of the probability distribution of a Gumbel random variable, so (( )) exponentials of exponential can be explain in terms of Poisson modules for arrival the counting underlying counting process. So, this module can be used to this is can be used to model the maximum earthquake ground acceleration in the time interval 0 to t in a given region, this again will be usual modeling tool and will you see more of this Gumbel distribution later in the course.

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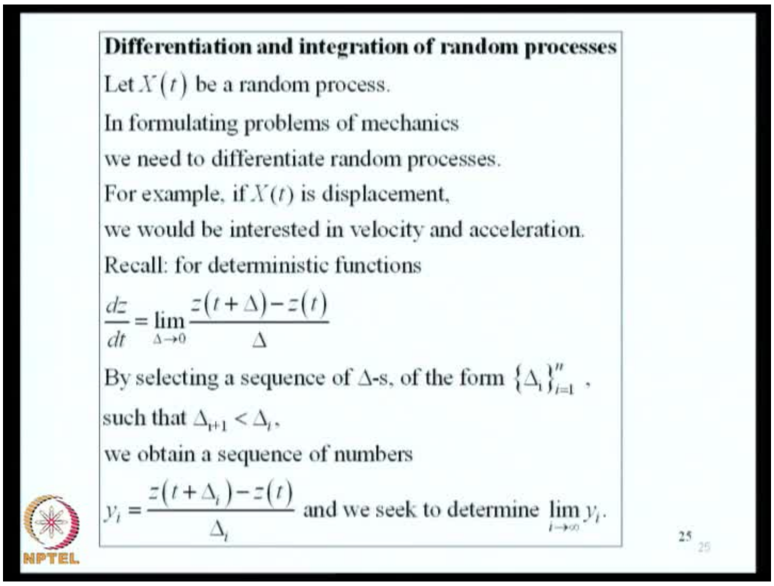

Now, we move on to certain mathematical description of random process and we ask the question what is meant by differentiation and integration of random processes? Let X of t be a random process and if you are dealing with problem of mechanics, if X of t is displacement I would be interested in dx by dt, which is velocity d square x by dt square which is acceleration, or in other words, do we need to develop the notion of a derivative of a random process, so, if you are looking at deterministic functions, suppose Z is a function of time and your looking by dZ by dt it is well known. Then, this given by limit of Z of t plus delta minus Z of t divided by capital T.

So, how do actually find dZ by dt, we actually form this ratio by selecting a sequence of delta S in the form delta 1, delta 2, delta 3, n and so that delta i plus 1 is less or equal to delta i and we get a sequence of numbers and we seek to determine limit of this sequence of numbers as i becomes infinity a sequence of numbers say x 1, x 2, x 3, x n we say that it converge this to x if for a given epsilon, I can define in n of epsilon. So that if I look at numbers beyond that n of epsilon, the different between those numbers and the x is less than epsilon that is a definition of conversion of sequence of numbers.

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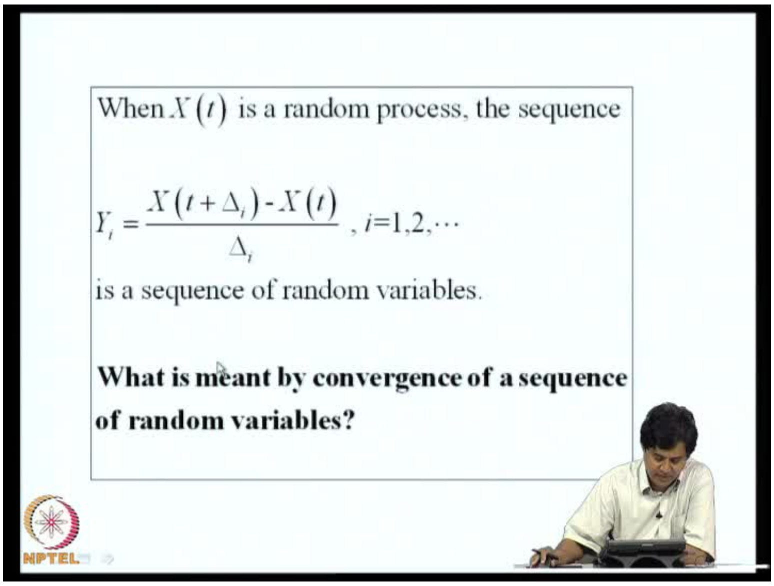

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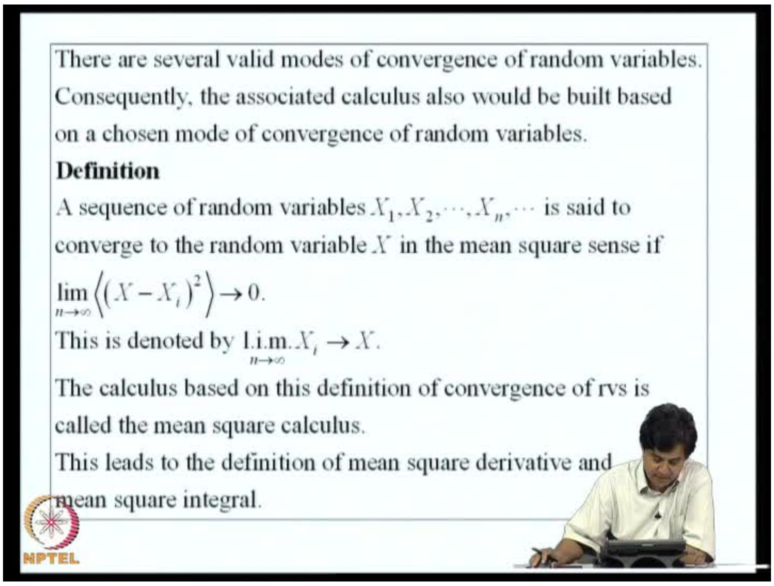

So that underlies is the definition of a derivative of a function. Now, if X of t is a random process, this sequence X of t plus delta i minus X of t by delta i is a sequence of random variables. So, what exactly is meant by convergence of a sequence of random variables, recall random variables is a function. So, what is meant by convergence of sequence of random variables. There are different notions of convergence that one can use, for example, these random variable y 1, y 2, y 3, y n are associated with the respective probability distribution functions. So, if corresponding to Y i, I have the i th probability distribution function, so, when I consider a sequence of a random variables, I consider a sequence of probability distribution functions. So, we can say that sequence of random variables Y i converged another random variable Y, if this sequence of probability distribution functions converge to the probability distribution function of Y then we say that we are taking about convergence in distribution, but the most useful notion of convergence is known as means square convergence and that is defined here.

So, a sequence of random variables X 1, X 2, X 3, X n is said to converge to the random variable X in the mean square sense, if the expected value of X minus X i whole square which is now a deterministic number it is a sequence of deterministic number go to 0 as n becomes large.

So, now the notion of convergence is associated with an expected value and that two expected values of the difference between X and X i mean square difference X minus X i is a random variable and I am considering mean square value; so, for any given delta there will be sequence of deterministic numbers and as n tends to infinite, this sequence go to 0, we say that this sequence of random variable converge to X in the mean square sense. This is denoted by l dot i dot m as n tends to infinite X i is X this has to be read as limit in the mean square sense.

The calculus that is based on this definition of convergence of random variables is called the mean square calculus, because this the notion of convergence underlies the definition of derivative and so also similarly the notion of an integral, depending on which mode of convergence that you actually imply in defining the your notion of convergence of sequence of random variables, we can have different types of calculus. so, one that we follow is the mean square calculus so this leads to the defined of mean square derivative and mean square integral so what they are.

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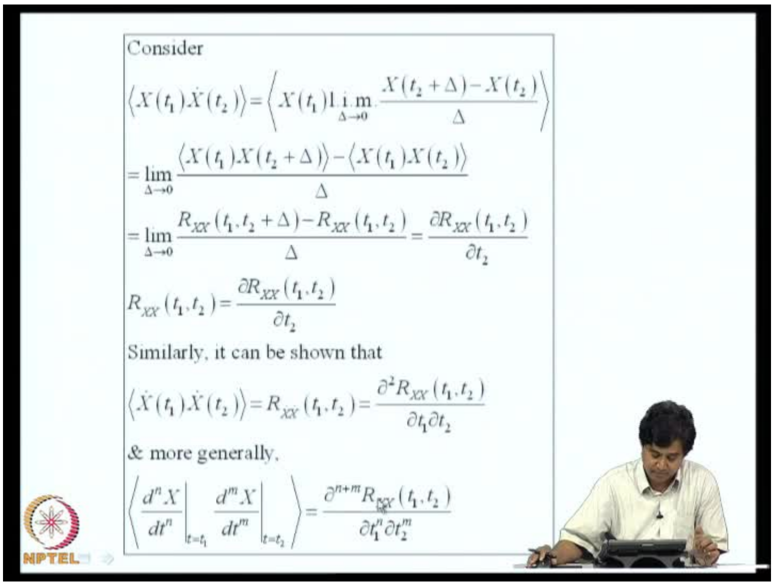

Now, let us consider expected value of X of t 1 multiply by the X dot of t 2 this dot is in the sense of limit in the mean square. So, X of t 1 is limit in square as delta going to 0 X of t 2 plus delta minus X of t 2 plus delta.

The expected value is the deterministic quantity. so, if I apply the limiting operation on the expected quantity it is a traditional limiting value it is not l dot i dot m. So, under certain condition this limiting operation and this limiting operations can be interchanged and assuming that such we can take such liberties this can be written as limit delta going to 0 X of t 1 X of t 2 plus delta minus this for delta.

Now, this is sequence of deterministic numbers. So, this is usual, you know sequence and limiting operation is a traditional limiting operation, because we are now dealing with deterministic quantities. So, what is this? This is nothing but auto correlation evaluated t 1 and t 2 plus delta minus this and divided by delta as delta goes to 0. This is nothing but the partial derivation of auto correlation of X with respective t 2 that is dou R XX dot t 1, t 2 is nothing but dou R XX t 1, t 2 dou t 2.

Similarly, if you want the auto correlation of X dot process R X dot X dot t 1, t 2 X dot of t 1 X dot of t 2. We can show that this is nothing but the second derivative of auto correlation of X X is the parent process X dot is derivative and is derivative with respect to t 1 and t 2 that means through this formulation, where able to derive auto covariance of derivative processes in terms of auto covariance of parent processes. In a more general formulation, if we are considering auto covariance of nth derivative and mth derivative, we can prove that this is given by this; I leave this as a exercise for you to tack out.

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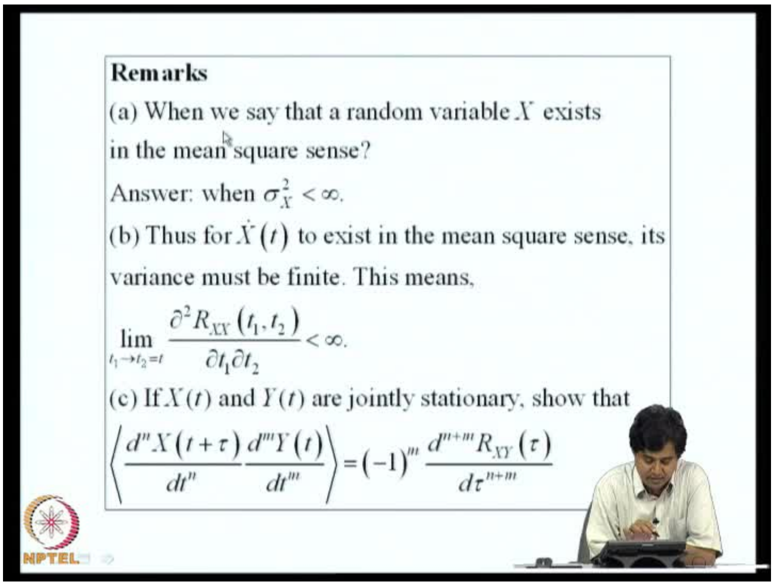

Now, we can now ask the question, when do we say that a random variable X exists in the mean square sense? The answer is the variance should be bounded. Thus, when do we say derivative of a random processes exist. So, thus for X dot of t to exist in the mean square sense its variance must be finite. That means limit as t 1 goes to t 2 goes to t the derivative of parent auto covariance function should be bounded, if this condition is satisfied then we can differentiate X of t in mean square sense that is the condition for difference ability of X of t.

Now, if we consider two random processes, I will come to this shortly and if we want, now the covariance between nth derivative of X and mth derivative of Y. We can show that the condition this is given by this, I will come to this shortly in greater detail.

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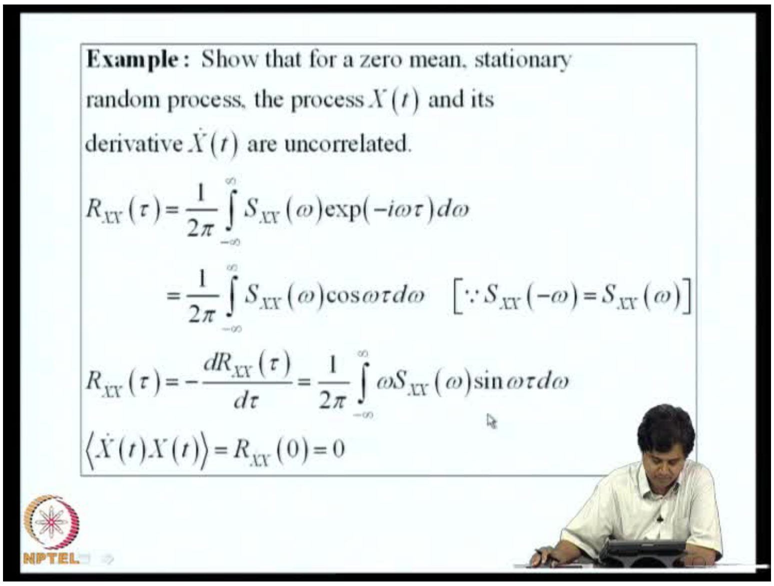

This stationary random processes X of t has an important property and this is an exercise to prove that namely show that for a zero mean stationary random processes. The process X of t and its derivative X dot of t are uncorrelated, how do we consider that we start by writing the auto correlation function since mean is 0 auto correlation and auto covariance are synonyms is given in terms of the power spectral density as shown here. So, using again the symmetric properties of power spectral density, I can replace a complex exponential by cosine function and we will look at this. Now, if I differentiate with respect to X time t that is if I consider R X dot X we have this relation.

So, if I now differentiate this with respect to tau I get this cos omega tau, becomes sin omega tau and this omega and it is the minus sin here and at the same time instead means is tau must be equal to 0. So, this will be R X dot X tau is always 0, because this again odd function, so this will be 0.

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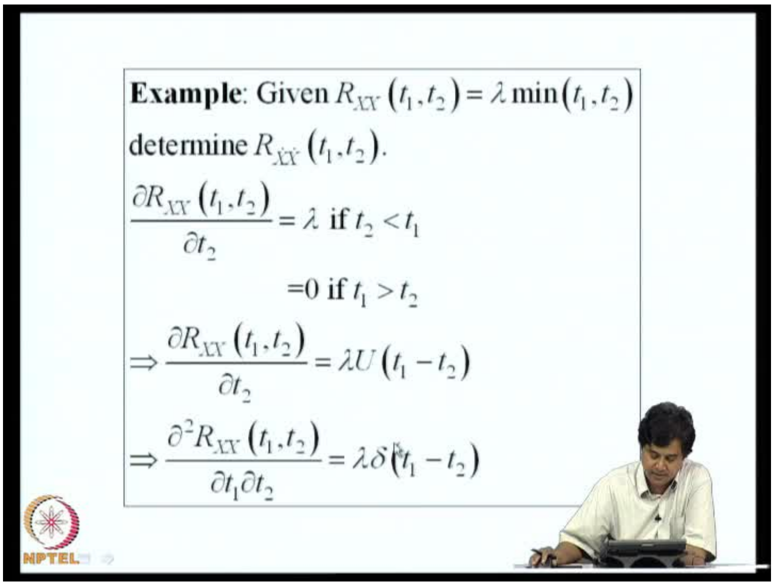

Let us consider another example, where R XX of t 1, t 2 is lambda into minimum of t 1, t 2. So, this function is not itself differentiable, so what, how will you handle this? So, how do you get the auto correlation of the derivative processes? Now, we know, now if I differentiate this once with respect to t 2 if t 2 is less than t 1 the derivative will lambda otherwise it is 0. So, I can write this using, the U side step function as lambda into U of t 1 minus t 2.

Now, if I differentiate this I can that we know the derivative of U side step functions is nothing but direct delta function using these generalize functions, I can write down the covariance of X dot of t. Where X of t has this auto correlation function, we will show that this is this. The process is known as white noise process, I will elaborate this shortly

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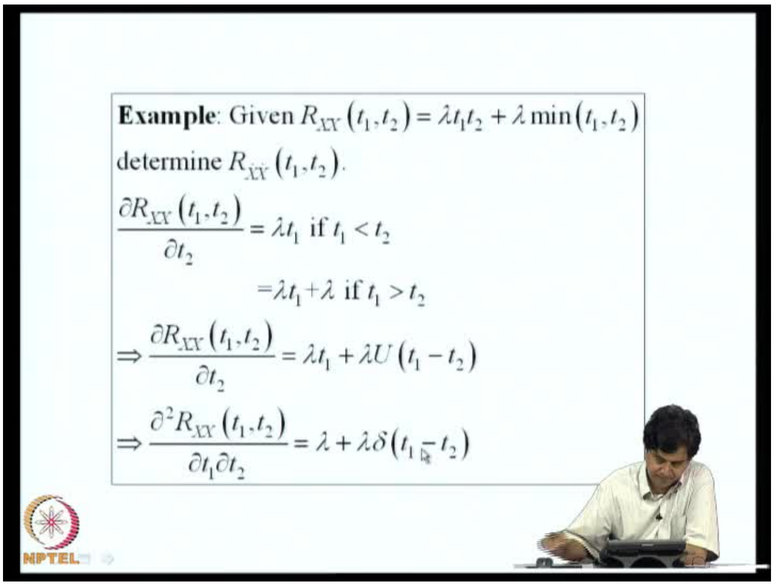

Similarly, if we have now R XX of t 1, t 2 is lambda t 1, t 1 t 2 plus lambda minimum of t 1, t 2 is a simple exercise to show that again using step functions and direct delta function, we can show that the derivative process will have this covariance structure.

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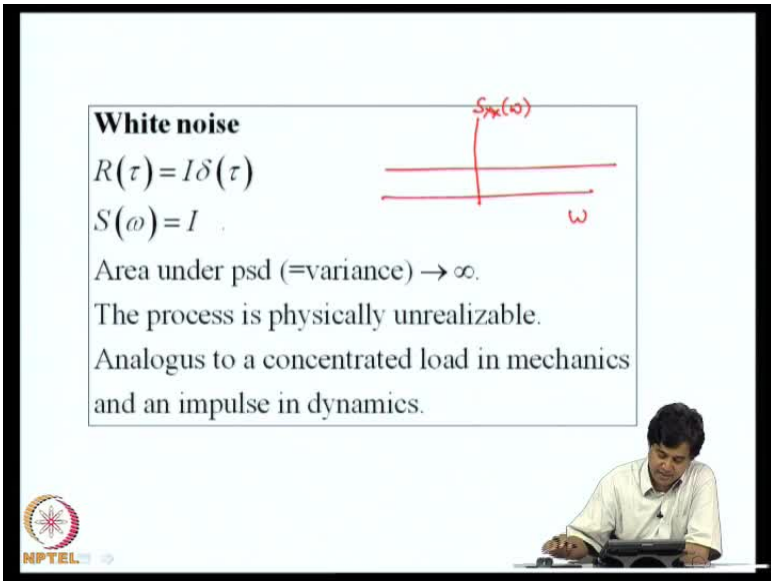

So, we define a random process, whose auto covariance function is a direct delta function to be a white noise process, if we consider its power spectral density function it is a constant. So, what is that we are looking at, so, this is the power spectral density function of a white noise process, what will be the area under this function it is unbounded. So, this process is not physically realizable. This is called white because the spectrum is constant across all frequencies just like in white light. White light has contribution from all colors in same proportion. Similarly, the total average power here they receive equal contribution from any frequency band. So, in that sense it is white so it is a flat power spectral density function.

Now, the notion of a white noise in random process modeling is analogous to the notion of a concentrated force in mechanics in statics or a notion of an impulse in dynamics a concentrated force is an idealization. Whenever we used a concentrated force as a model there will be an integration that will be pending, whenever use impulse as a model in terms of using direct delta functions and integration would be pending, the same sense here. Whenever we model a random process as a white noise process, there will be filtering or passing through another differential equation an integration would be pending right that has to be (( )) in mind. So, white noise is a very powerful modeling tool just as a notion of a concentrated force is a very powerful modeling tool in mechanics.

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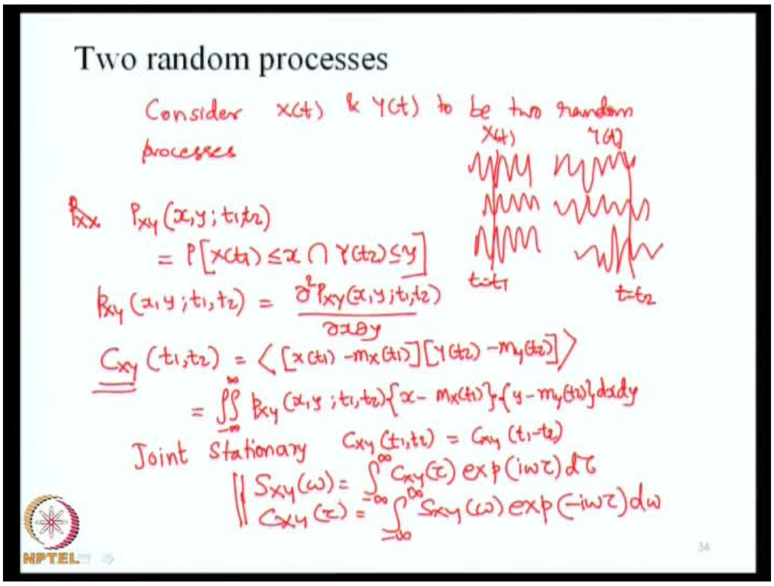

We will see again, more of this in due course. So far, we have been talking about single random process and had brief occasion to talk about two random processes. Now, let us formalize the notion of two random processes. So, let us consider X of t and Y of t to be two random processes, we know what constitute complete description of X of t, and what constitute complete description of Y of t, but what constitutes complete description of X of t and Y of t.

Clearly, we have to provide join probability distribution functions, suppose you fix a time t equal to t 1. So, X of t is an ensemble suppose, this is a sample of X of t and suppose these are samples of y of t, suppose I take time instant t equal to t 1 and t equal to t 2 here, X of t 1 is a random variable Y of t 2 is a another random variable. So, I can now consider probability of XY x, y t 1, t 2 this can be defined as probability of X of t 1 less or equal to x intersection Y of t 2 less or equal to y.

This random variable is originating from X of t. This random variable is originating from the ensemble Y of t. so, I can define associated with this the notion of the joint probability density function between X and Y, which is the derivative of the distribution function dou y. So, this argument can be repeated for more than two time instants I can take 2 time instants here and 5 time instants here or another words, I can draw certain random variables from X of t certain random variable from Y of t and consider their joint distribution function and joint density function.

So, this collection of random variables which originate from both X of t and Y of t, if you are able to provide their joint probability density functions for any choice of this number of this random variables and any choice of time instant from, which they originate for able to provide joint descriptions, we say that we have provided complete description of X of t and Y of t.

Now, what are others simpler description of two random processes, we can talk about, cross covariance of X of t and Y of t 2 so t 1, t 2 that means, this is expected value of X of t 1 m X of t 1 y of t 2 m Y of t 2. So, this is defined as minus infinity to infinity p XY x, y t 1, t 2 of what x minus m X of t 1 y minus m Y of t 2 dx dy.

So, I was using the term auto covariance, when I was describing a random process. Now, the reason for the prefix auto is now apparent. This quantity is called a cross covariance function. The word cross denotes that one of the random variables namely, X of t 1 is drawn from X of t and other y of t 2 is drawn from Y of t it is between two random variables originating from two different random processes. So, that is why it is called cross covariance.

We introduce the notion of stationarity. So, we can now extend the question when do we say X of t and Y of t are stationary, all when do is say X of t and Y of t are jointly stationary for that X of t should be stationary, Y of t should be stationary; in addition they should be jointly stationary also by that what is meant is C XY of t 1, t 2 must be equal to C XY of t 1 minus t 2, it is not enough if X of t stationary on its own right and Y of t stationary on its own.

The joint descriptors, like joint probability density function or the joint moments like cross covariance function should also display that property. So, X of t and Y of t in their own rights can be stationary but jointly they can be non-stationary.

Now, associated with jointly stationary random processes, we have now defined cross covariance function and we can also define the associated cross power spectral density function. So, that can be defined as XY of omega as minus infinity to plus infinity C XY of tau exponential i omega tau d tau. And this is related to the pair minus infinity to infinity S XY of omega exponential minus i omega tau d omega.

So, we can defined the Fourier transform pairs C XY tau and S XY of omega in the next class, will continue with the discussion on two random processes and we will clarify properties of cross power spectral density function and cross covariance function.