Stochastic Structural Dynamics

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Module No. # 04

Lecture N0. # 15

Random Vibrations of MDOF Systems - 3

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Lecture-15

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We have been discussing random vibrations analysis of discrete multi-degree freedom systems. So, we will continue with that discussion and we will be concluding this discussion in this lecture, and we will begin discussion on random vibrations of continuous systems.

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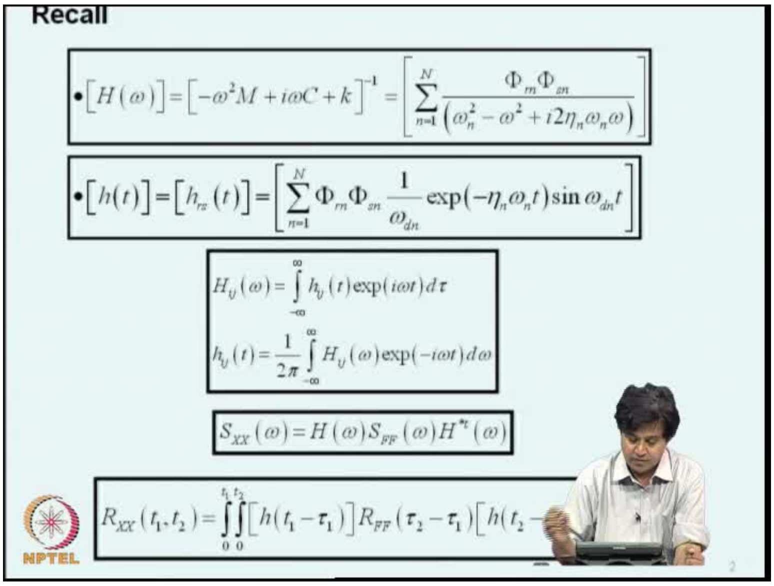

So, what we have done till now is that, we have characterized the multi-degree freedom system through frequency response function and impulse response function; these functions now become metrics. So, for a n degree freedom system, the metrics of complex frequency response function is a n by n square matrix; it can be expressed in terms of structural matrices as shown here or in terms of the natural frequencies, mode shapes and model damping values as displayed here.

Similarly, the impulse response function here becomes a matrix - a square matrix - of size n by n and this is determined in terms of the mode shapes, the damp natural frequencies, model damping as shown here. The elements of this complex frequency response function matrix and impulse response function matrix are shown here and we saw that they form a Fourier transform pair; this is to be expected, because we saw similar relations being applicable for single degree freedom systems. The input, output relation in terms of a power spectral density function is shown here; S FF omega is the matrix of power spectral density functions of the input; input is modeled as a stationary vector random process with 0 mean; S FF omega is the power spectral density function; H of omega is the complex frequency response function matrix.

Similarly, the matrix of covariance of the vector X of t 1 and X of t 2 is related to the matrix of the input auto correlation function matrix through this relation, where here h is the matrix of impulse response functions, t is the transposition, star t is the transposition with conjugation. So, this is what we have discussed in the previous lecture.

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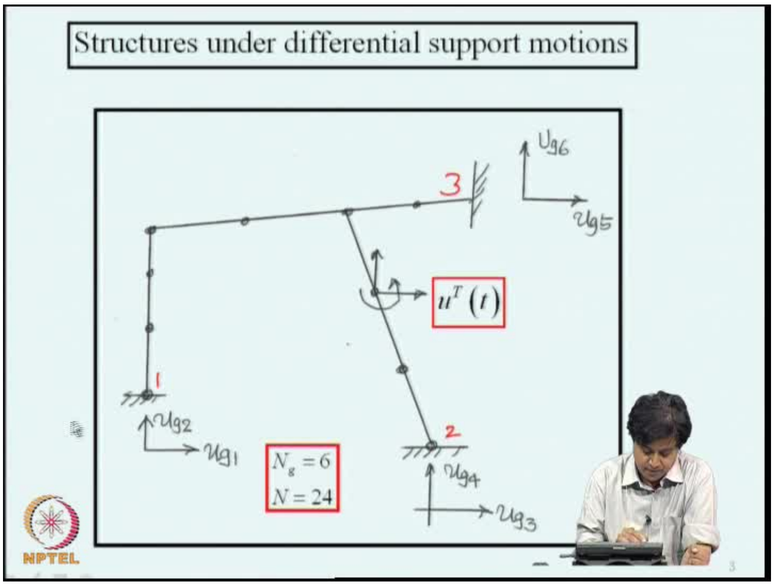

We began discussing about the action of earthquakes on extended structures and we started talking about structures under differential support motions. These types of structures are typically found as piping structures in say industrial power plants or also they are also associated with say long span structures like bridges, dams and so on and so forth. The main property here is that the support displacement here varies in space that would mean for this structure that is shown here, there are three supports; here, this is one support, this is other support and this is third support. And in this illustration, we are considering the action of two support displacements at each of these supports; so there are six support displacement components.

And if we use, for example, finite element formulation to analyze this frame structure at every node, we can have three degrees of freedom and this we call as the total displacement of the super structure. So, the collection of these degrees of freedom is encapsulated in the vector u T of t; N g, we call is the number of support displacements, for this particular illustration, it is 6, and N is the super structure degrees of freedom and in this particular instant, it is 24, because there are 8 nodes 1, 2, 3, 4, 5, 6, 7, 8 and at each of this node, there are three degrees of freedom. So, we have N is 24.

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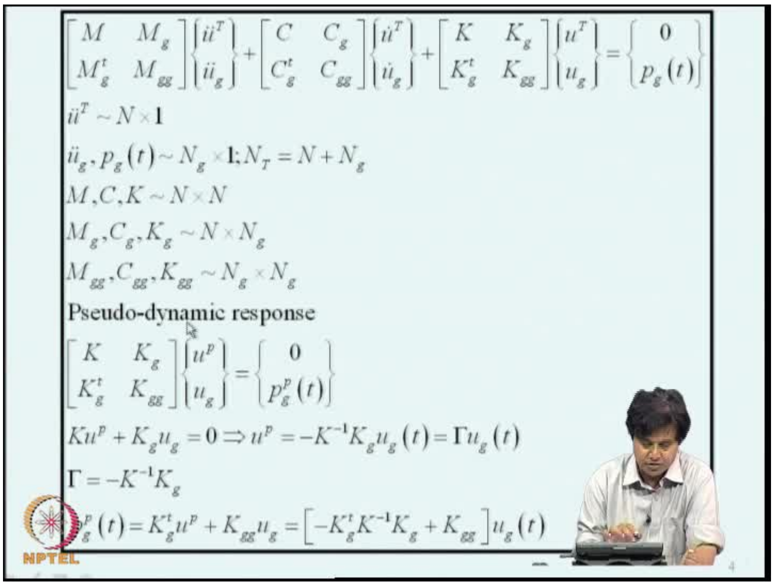

So, we can write the equations of motion. If you write the global equilibrium equation here, the degrees of freedom can be partitioned as the super structure degrees of freedom and the support displacement, these are applied support displacements; based on this partitioning of the displacement vector, the associated structural matrices can also be partitioned so M M g, M g transpose M g g and similarly damping and stiffness matrices. P g of t is the reactions due to the support motions at supports and u T is n cross 1; u g P g are all N g cross 1, and N T which has the size of these matrices is N plus N g; M C K here are square matrices, whereas M g, C g, K g are rectangular matrices and in the analysis of this type of system of equations, we first find the so called pseudo dynamic response by considering only the static behavior of the system.

So, if we consider now the static behavior of the system, we call the response as pseudo static response so that super script P denotes the pseudo static or pseudo dynamic response and the equilibrium equation is as shown here. So, this is the response of the structure simply due to the differential support displacements; there is no inertial actions or energy dissipation involved here; it is purely a static behavior, but still this this equilibrium equation has to be analyzed at every time t therefore, the word pseudo dynamic is used to convert this fact.

So, if we now write the equation corresponding to the first row, I get K into u P plus K g into u g is equal to 0; from this I can express u P in terms of K, K g and u g as shown here and this matrix K inverse K g minus K inverse K g I call it as capital gamma and this is this can be viewed as a kind of a influence matrix. P g of t superscript P is the pseudo static component of the reaction that can be obtained from the second row of this equation K g transpose u P plus K g g u g is P g t so this has to be found after we solve for u P. So, this gives the pseudo static reactions; this gives the pseudo static or pseudo dynamic displacements.

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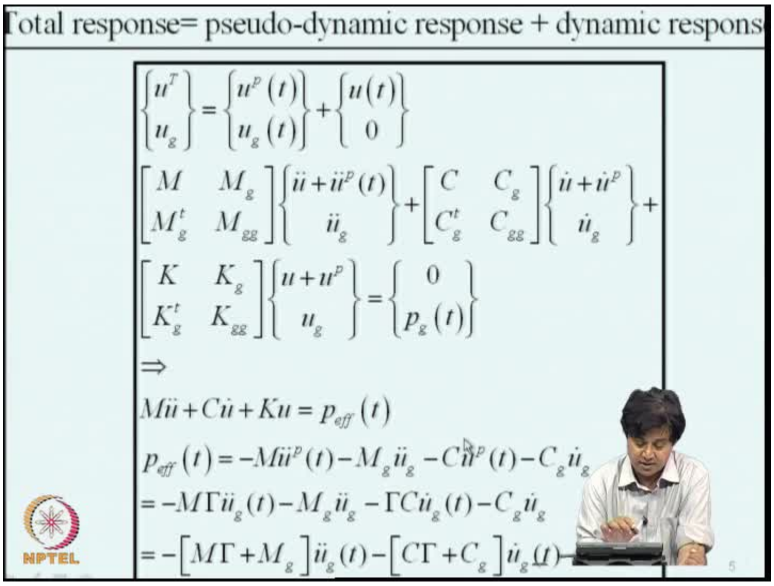

Now, we split the total response into a pseudo dynamic component and a dynamic component. So, moment we substitute this into the governing equations, we can derive now the equation for u of t which is the unknown; u P of t has already been determined; so the unknown here is u of t. So, if we now look at this equilibrium equation, wherever there was u t, I am writing u double dot plus u P double dot as shown here.

Now, if I write the equation for the first row, I can get M u double dot plus M g into u g double dot and m into u P double dot; C into this term plus C g into this term; similarly, K into this term, K g into this term will be, I can designate that has P effective t. So, this P effective t can be returned in terms of the pseudo static response and support acceleration as shown here acceleration velocities. Now, for pseudo static component of the response, I write u P is minus gamma u g double dot or plus gamma u g double dot; if I do that and rearrange the terms, the effective force gets expressed in this form. So there is a kind of mass matrix, which multiplies acceleration; there is a damping matrix, which multiplies the velocity; gamma is a kind of influence matrix that arises in computing pseudo dynamic response.

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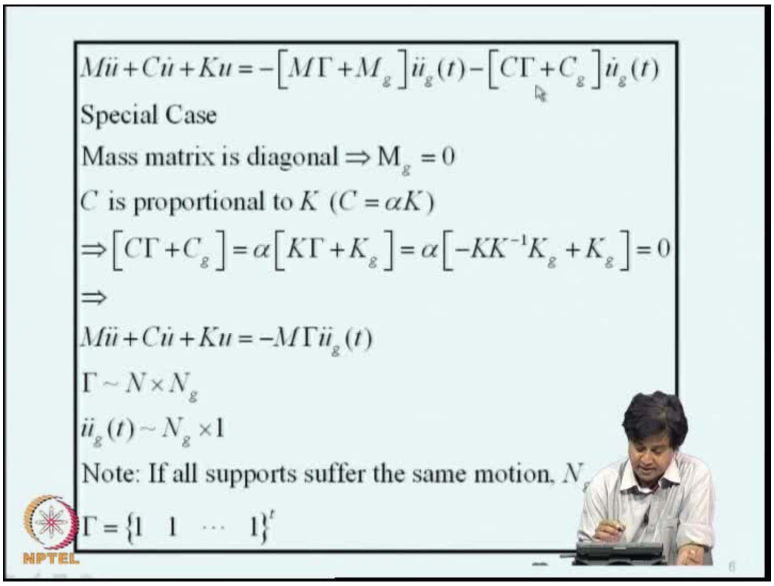

So, we can just examine the nature of this equation; suppose if M is diagonal that would mean M g is 0, because M g is of diagonal component of the mass matrix it is 0 and suppose C is proportional to stiffness matrix, in that case I can write this C gamma plus C g as say some alpha into K gamma plus K g; now for gamma if I write minus K inverse K g we will see that the contribution from the C gamma plus C g becomes identically equal to 0 and the contribution from this second term to the effective force becomes 0, and the contribution from acceleration terms now read as minus M gamma u g double dot of t u, since u g is now N g cross 1 vector, gamma will be a n cross N g matrix.

Now, if all supports receive identical support displacements, then N g will be equal to 1 and u g double dot t will be as scalar function and gamma would become simply a vector of units and this equation reduces to the case of uniform support motion that we discussed in the earlier lecture. So, there is a consistent consistency between what we are discussing here and what was discussed before.

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Random vibration analysis in frequency domain

$$M\ddot{u}+C\dot{u}+Ku=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)=p(t)$$

$u_g(t)\sim N_g\times 1$: vector of stationary random process with zero mean and PSD matrix

$$S_{gg}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_{gT}(\omega)U_{gT}^{*t}(\omega)\right\rangle$$

$$p(t)=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)$$

$$P_T(\omega)=\omega^2\left[M\Gamma+M_g\right]U_{gT}(\omega)-i\omega\left[C\Gamma+C_g\right]U_{gT}(\omega)$$

$$=\left[\omega^2\left[M\Gamma+M_g\right]-i\omega\left[C\Gamma+C_g\right]\right]U_{gT}(\omega)$$

$$P_T^{*t}(\omega)=U_{gT}^{*t}(\omega)\left[\omega^2\left[\Gamma^t M+M_g\right]+i\omega\left[\Gamma^t C+C_g\right]\right]$$

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Now, let us now turn our attention to the problem of random vibration analysis and we will begin by considering the analysis in frequency domain for steady state oscillation. This is a simpler problem in comparison to time domain analysis, where initially there will be a transition non-stationary response and if there is a stationarity, it will reach that state as time becomes large; if excitation is non-stationary, of course, then we will have to deal with the problem in time domain.

Now, we will assume that u g of t is a vector of stationary random process with 0 mean and PSD matrix given by this - S g g of omega - this is the PSD matrix for support displacement. Now, if you see the details of this forcing on the right hand side, we do we have velocity and acceleration terms; so if we start modeling u g of t in terms of displacement power spectral density, then we have to deduce the power spectral density for acceleration and velocity and we are ready with that; we know what it is. So, suppose P of T is this; now if you now take the Fourier transform of a truncated version of P of T, this is given by this; here, now we get Fourier transform of the displacement, although we have acceleration here; I will start working with displacements.

Now, this is the Fourier transform of the effective force and if its conjugate transpose is considered, we get this; we have to take transposition and conjugation, we get this expression; now if you want the power spectral density, we have to multiply this with this and apply this take an expectation and apply this limiting operation.

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$$S_{pp}(\omega) = \lim_{T\to\infty} \frac{1}{T} < \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$
$$U_{gT}^{*t}(\omega)\left[\omega^2\left[\Gamma^t M + M_g\right] + i\omega\left[\Gamma^t C + C_g\right]\right] >$$
$$S_{pp}(\omega) = \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]$$
$$S_{gg}(\omega)\left[\omega^2\left[\Gamma^t M + M_g\right] + i\omega\left[\Gamma^t C + C_g\right]\right]$$
$$\Rightarrow$$
$$S_{UU}(\omega) = H(\omega) S_{pp}(\omega) H^{*t}(\omega)$$

$$\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1} = \left[\sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$$

So, if you now do that, S pp of omega is limit t to infinity 1 by t of P T of omega multiplied by P T star transpose omega; if I do that and carry out these simplifications, I get the expression for S pp of omega in terms of the power spectral density of vector of support displacements, the structural matrices and the influence matrix capital gamma.

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Random vibration analysis in frequency domain

$$M\ddot{u} + C\dot{u} + Ku = -\left[M\Gamma + M_g\right]\ddot{u}_g(t) - \left[C\Gamma + C_g\right]\dot{u}_g(t) = p(t)$$

$u_g(t) \sim N_g \times 1$: vector of stationary random process with zero mean and PSD matrix

$$S_{gg}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle U_{gT}(\omega) U_{gT}^{*t}(\omega) \right\rangle$$
$$p(t) = -\left[M\Gamma + M_g\right]\ddot{u}_g(t) - \left[C\Gamma + C_g\right]\dot{u}_g(t)$$
$$P_T(\omega) = \omega^2\left[M\Gamma + M_g\right] U_{gT}(\omega) - i\omega\left[C\Gamma + C_g\right] U_{gT}(\omega)$$
$$= \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$
$$P_T^{*t}(\omega) = U_{gT}^{*t}(\omega)\left[\omega^2\left[\Gamma^t M + M_g\right] + i\omega\left[\Gamma^t C + C_g\right]\right]$$

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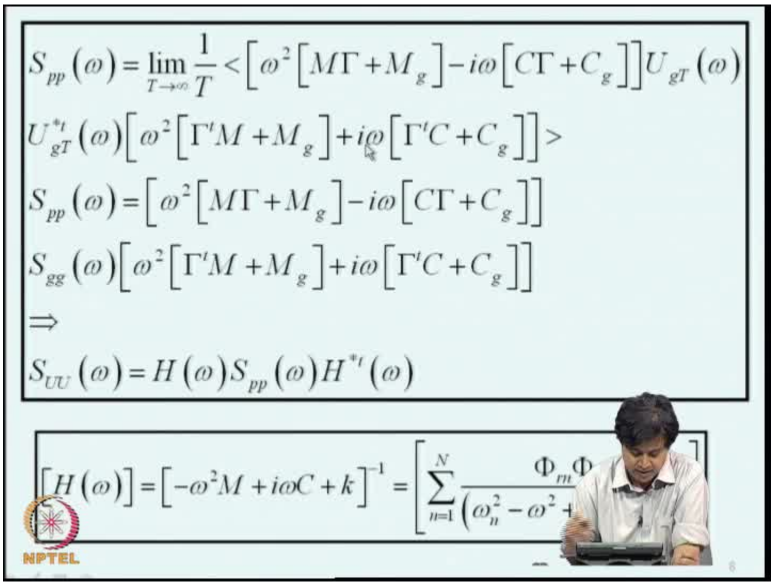

Now, moment I know the power spectral density of this quantity p of T; I can now use the standard input, output relation in frequency domain; I get the output power spectral density function for u of T; this is a matrix given by h into S pp of omega, which is this long expression and H star transpose omega, where H of omega is of course this and in terms of M C K and in terms of modes, it is this.

So, these are reasonably straightforward exercise; so the only issue is you have to handle computation of power spectral density function for the effective force. You are given model for support displacement and you have to deduce the power spectral density function for the effective force and moment that is done, the standard input output relations take over and you get the solution.

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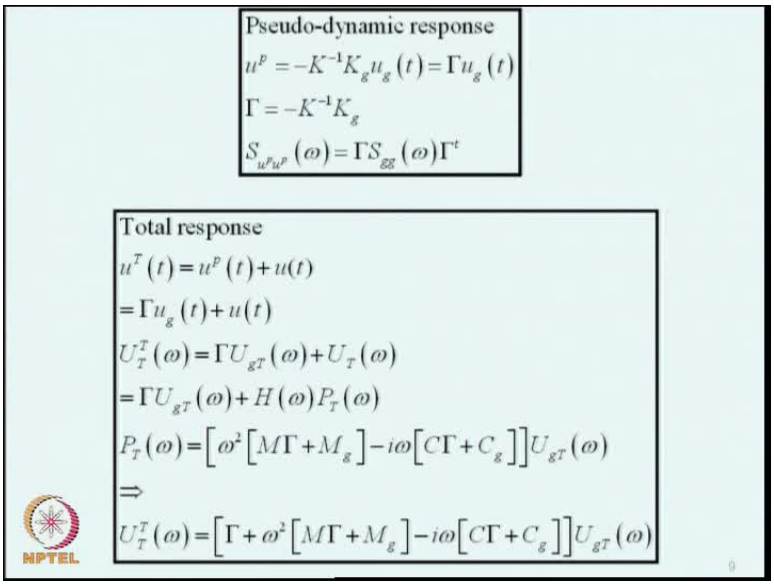

Suppose you are looking at only pseudo dynamic response u p of t is given by gamma u g of t therefore, the power spectral density function of the pseudo dynamic response component is simply gamma S g g of omega gamma transpose and this is this. So, gamma is a here is real value; therefore, there is no need to conjugate and do that; so this is the power spectral density of a pseudo dynamic component. So, total response can be written as pseudo dynamic component plus dynamic component; so this is gamma u g of t plus u of t. Therefore, if you now take that truncated Fourier transform of the total response, we get this and if you now express U T of omega in terms of P T of omega, I get P T, where P T of omega is already determined; I can derive now the truncated Fourier transform of U T T of omega and that is given by this expression in terms of u g T of omega.

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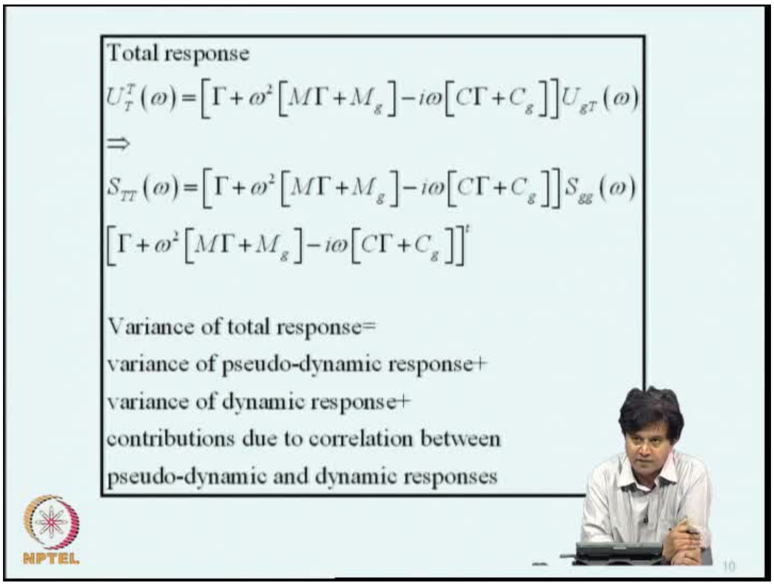

With this, we ready to find the power spectral density function of the total response, suppose if I call it as S T T of omega, few matrix manipulations will take us to this quantity. You can carefully design that this total power spectral density the power spectral density function of the total response would obviously have a component due to pseudo dynamic response and dynamic response and contribution from correlations between dynamic and pseudo static response. Thus suppose if you compute variance of the total response, it will have component from variance of pseudo dynamic response, variance of dynamic response plus a contribution due to correlation between pseudo dynamic and dynamic responses.

So, in the design of piping structures in power plants, it is necessary to compute these three quantities separately; so we need to know that formulary for to achieving that goal.

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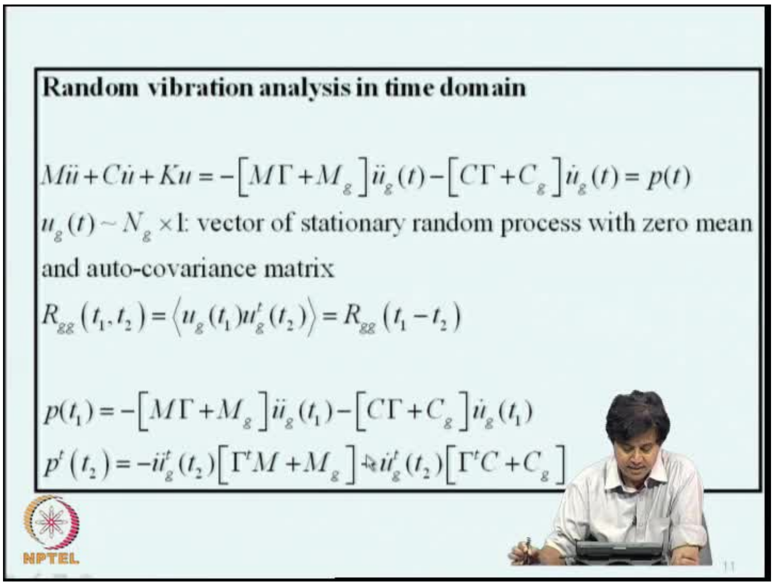

Now, how about the analysis in time domain? So, we are back to this equation, where P of t is a effective force as before; it is in terms of u g double dot of t and u g dot of t, and u g of t is vector of stationary random process with 0 mean and auto covariance; we will now write it in terms of R g g t 1 minus t 2 this could as will be (t 1, t 2) if it is non-stationary, but this is what we are using in this illustration. So, I have P of t 1 is this and P transpose of t 2 is transposition of this evaluated t equal to t 2 and I get this.

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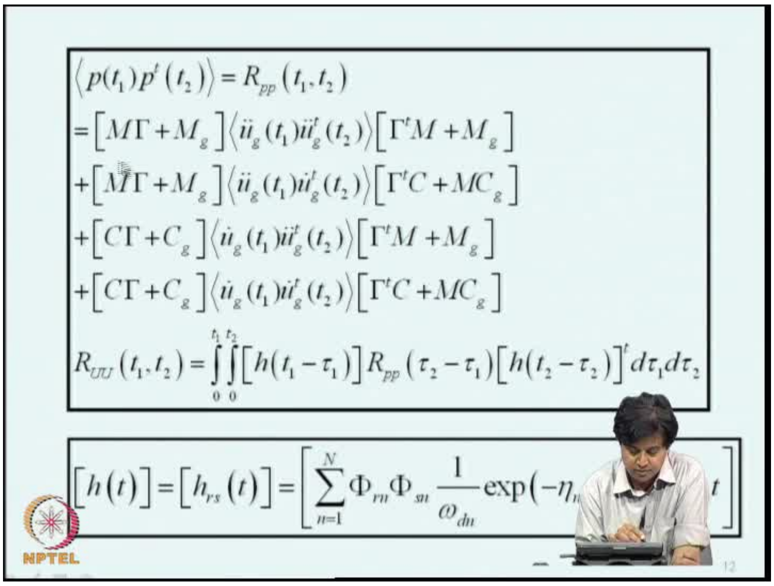

If I multiply these two and take expectation, I will get the covariance matrix for the vector random process P of t. So, if I do that, I get R p p of (t 1, t 2) and in terms of the structural matrices and cross covariance, auto covariance's between acceleration, support accelerations and velocities, etcetera we get this expression. If input is completely specified, it is possible to compute all these expectations and once the auto covariance of the input is given, we now resort to the time domain relation for output auto covariance in terms of input auto covariance and we get this equation H of t is a matrix of impulse response functions expressed in terms of normal modes and natural frequencies and model damping as here; so if you plug that in, you have to carry out the requisite number of integrations and you will be able to characterize the response.

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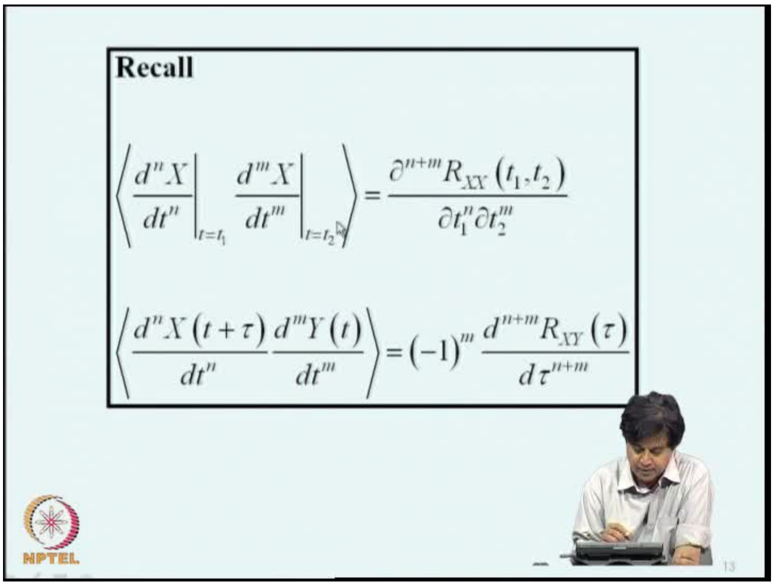

Now, how do we compute these expectations? For example, you have to compute u g double dot of t 1 and u g dot of t 2, starting from auto covariance of u g of t. So, if you recall, we have shown that auto correlation between N th derivative of X of t and M th derivative of X of t is given by this and if the process is stationary, we get this; this can as well be generalized to find out this type of covariance.

So, these are the matrices; each element of this can be for determined following this rule. This rule, for example, is applicable for two random processes X and Y, here we can as well replace X by Y, and this will become R X Y. So, if there are two random processes, we can still use one of these two; if you are in non-transition domain, you have to use this; if process is non-stationary you have to use this; if process stationary, you could use the simpler version.

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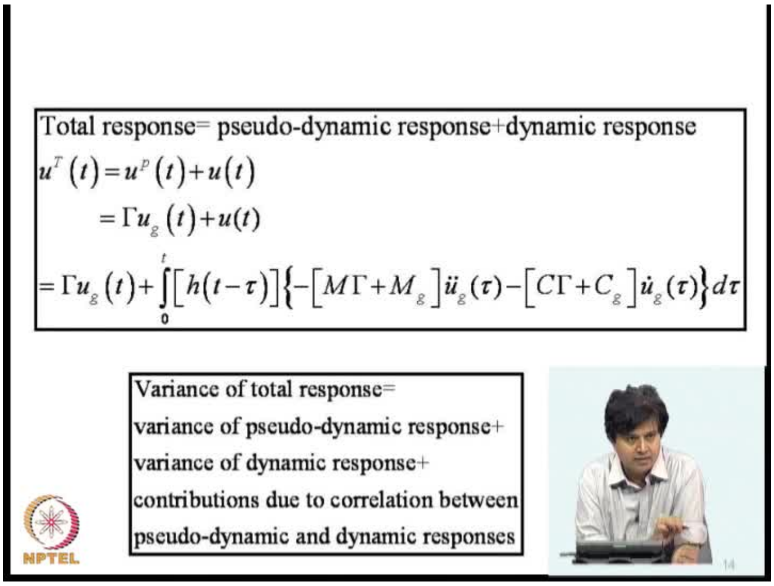

Again if you look at the total response, it has pseudo dynamic response and a dynamic response; the u P of t plus u of t- u P of t continues to be gamma times u g of t. Now, this u of t can be given in terms of the impulse response function matrix and this vector of effective force, in principal we have got the expression for u P of t and you can take expectation of that u t of t, you can take expectation of that you will get the mean; you multiply u t of t by its transpose and take expectation, you will get covariance of that and so in principal, this is evidently dual. Here, again we can see that variance of total response even in transition state will have a variance of pseudo dynamic response plus variance of dynamic response and contributions due to correlation between pseudo dynamic and dynamic responses.

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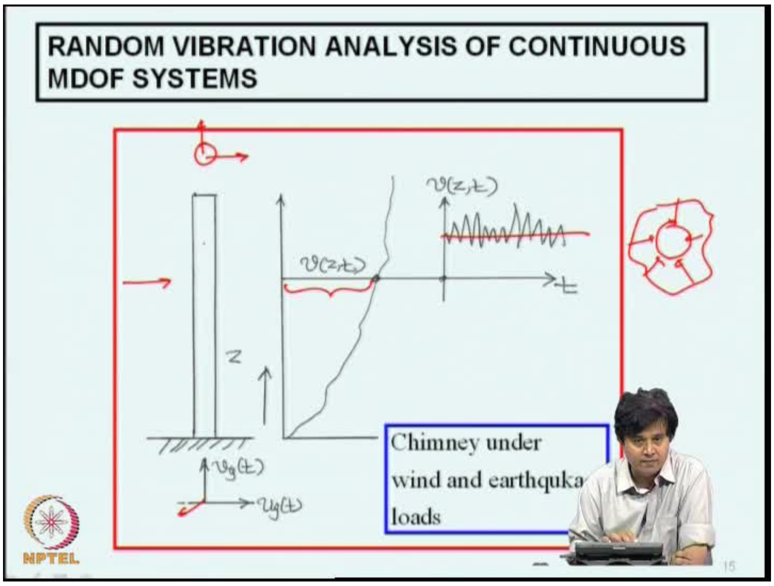

So, this in a way completes the analysis of multi-supported multi degree freedom systems, which are which are subjected to differential support motions. Now, we take up we move on and consider the problem of random vibration analysis of continuous systems. So, we are moving to a new topic, but many features of what we are going to discuss now are quite similar to what has been discussed for discrete multi degree freedom systems. For instance, the notion of nature coordinates can be generalized to continuous systems, the notion of frequency response functions and impulse response functions, matrices, here become so call Green's functions. So, all those concepts are easily extendable for continuous systems, but it is worth going through this exercise.

So, to start with, we can consider a situation, where a continuous system serves as an adequate model to model an engineering structure. For example, if you consider a chimney, it can well be modeled as a cantilever beam and if it is subjected to, say wind load, this is the profile of the wind velocity; at ground, the wind velocity is 0 due to the boundary layer effect and as we climb up, it reaches the atmospheric wind velocities and this this is again random in nature; so this is snapshot of the wind velocity at a given time T.

But on the other hand, if you look, if you trace the velocity of the wind at a given elevation here, as a function of time, it will look like this; this mean this mean corresponds to this value and the oscillations takes place here about this mean due to turbulence. Now, this structure is immersed in a flow like this; so the due to this flow, there will be a pressure field around the object and if you integrate this pressure field over this surface area, you get the an effective force and that can be resolved in the direction of the flow and perpendicular to that, we call that as this as drag direction; this as across wind direction.

Now, the calculation of this pressure field can become somewhat more involved, if this structure is flexible; in which case there will be an interaction between the vibration of this chimney and the pressure field around that and this directions also keep changing; there will be an effective drag direction and effective across wind direction. In any case, the force field around this chimney will be a function of square of this velocity and that will be a random process; that evolves in time as well as along the height of the chimney.

So, if you model this as a cantilever beam, we have to consider that dynamics of a cantilever beam our beam, subjected to an external force say F of say (Z , t), where F of (Z , t) is a random field which evolves in space as well as in time. Of course, the chimney can also be subjected to earthquake ground motion, although I have shown both these actions together; it is not implied that the occurring together, not necessary; we seldom considering this type of situation, it just for an illustration I am showing. In the event of an earthquake, this supporting point here will be subjected to say actually three components also you can consider, two horizontal components of support displacement and one vertical component of supports displacement.

So, these again are each component can be modeled as a random process so then we will have a continuous systems, where boundary conditions are time varying and their model the time variation are modeled as random processes. So, we need to consider if you are interested in earthquake response analysis, the behavior of the structure under random time varying boundary conditions and if you are interested in wind like situation, you have to consider the dynamics of say beams under external forces, which are random fields in space and time. So, we will consider how we can tackle with this problem, but before we get into the random vibration analysis, we will first review quickly a few important results from deterministic analysis of continuous systems.

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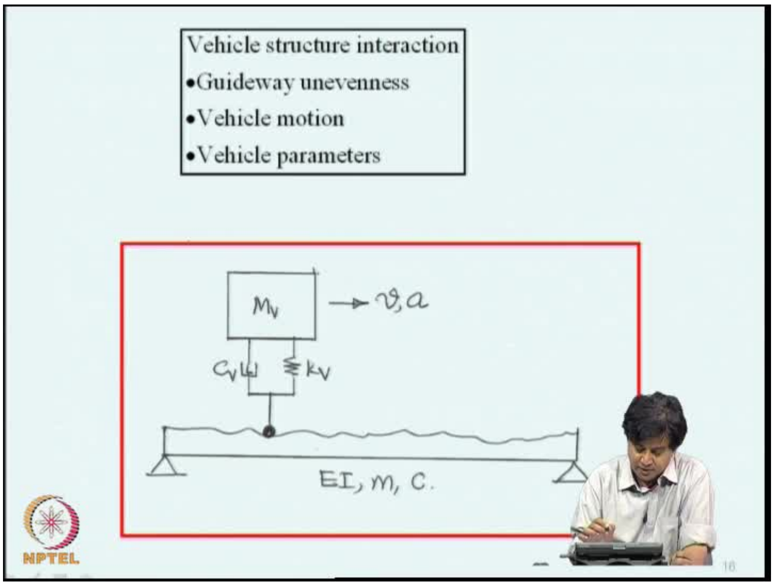

Another example is that of dynamics of a bridge like structure when a vehicle passes on the bridge, suppose this is a railway bridge; this can be viewed as a simple model for as a locomotive. Because of guide way unevenness, the passage of this locomotive and the bridge, there will be complex interaction between the regularities here and the property of this vehicle and the effective force that get transmitted - the wheel force that get transmitted - to the structure will be a random process and if this also another example of random vibration of a continuous system, where the bridge to a first approximation can be modeled as a beam.

Of course, this is a hybrid problem, where the vehicle is modeled as a discrete system and bridge is modeled as a continuous system; this is one of these archetypal models for studying vehicle structure interaction. In the context this problem, of course there can be other sources of uncertainties, for example, the vehicle velocity and acceleration could be changing, they can be random; the vehicle payload that is M, this mass itself can be random and of course, the damping and spring elements could also been uncertain in nature. Not only that, at any given time, there can be more than one vehicle on the structure; if it is the highway bridge, if you take the snapshot of the bridge aerial photograph of the bridge, you will see that vehicles will be randomly distributed in space within the bridge and there will be moving in random directions, if there are eight line traffic for example; the direction of direction and velocity of moment also will be could be random. So, it is a very complex loading system and if we model this traffic also as a random process, then we have additional complexities.

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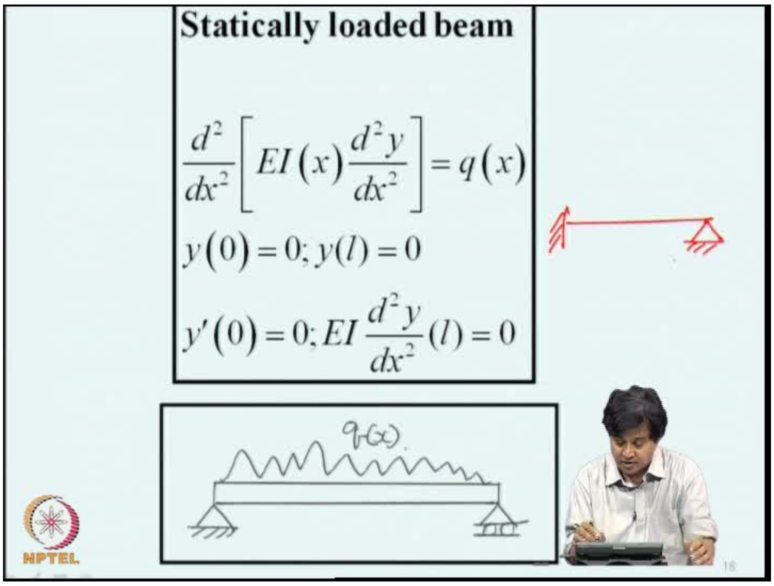

So, we will begin this discussion by conducting a quick review of dynamics of continuous system under deterministic excitations and I select Euler - Bernoulli beams as an representative of this continuous systems; this could as well be a actual vibrating rod or shear deferming soil layer, a plate or cell. So, we will begin this discussion now; we will first consider the equilibrium equation for a statically loaded beam, this if a beam property varying with respect to space, the beam structure considered is shown here; this E I is the function of x, and q of x is the external load, the equilibrium equation is d square by d x square EI of x d square y by d x square is q of x, and the specified boundary conditions here, this corresponds to not to this beam, it actually corresponds to a structure which is clamped here and hinged here.

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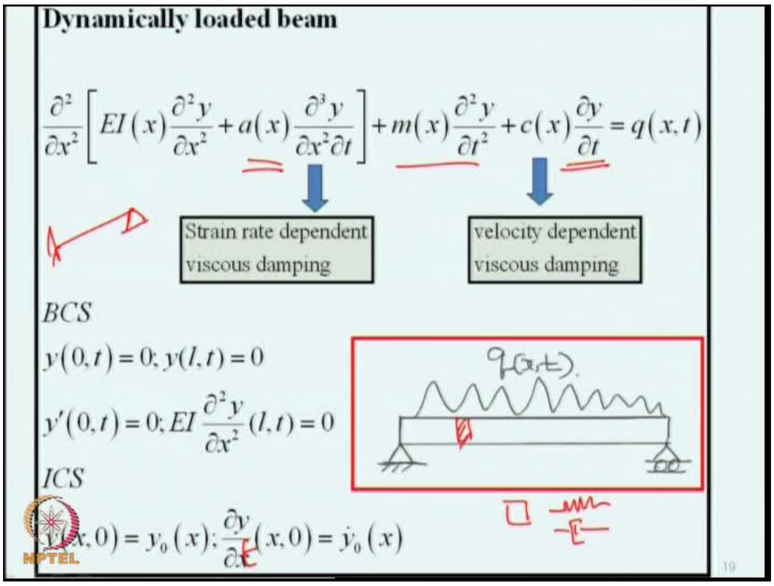

This is one example that I will be considering to illustrate the concepts; so we will proceed with this. Now, if the same structure is known loaded dynamically, that means, if this forcing function is a function of both x and time, in addition to the elastic forces, we also have now inertial forces and forces due to energy dissipation. So, this is the inertial force and the there are the damping force are represented at two places; this is one term and there is a another term here; this is a damping force, which is proportional to the velocity of the beam; on the other hand, this is the force, which is proportional to the time derivative of the bending strain.

So, the general damping model could have terms proportional to displacement and proportional to proportional to velocity and proportional to velocity of the strain; so we have included for the sake of illustration these things. Again for this structure, that is that of a proper cantilever, the boundary conditions or displacement is 0, here and here slope is 0 and bending moment at the hinged end is 0 and the structure could start with certain initial displacement and certain initial velocity.

So, this is their field equation; this is valid for any boundary conditions, it can be a simply supported beam, free beam, fix beam, etcetera the field equation would remain the same. What would distinguish different beams types would be the boundary conditions and together with the initial conditions, this forms the governing equation of motion. So, this is now a partial differential equations -a fourth order partial differential equation with y of x as the dependent variable; x and t as independent variables; so we need to develop a strategy to solve this equation.

So, we will try to now explore based on our experience with dealing with discrete multi-degree freedom systems, we will try to see now if we can perform a transformation of dependent variable and introduce certain new generalized coordinates such that in the new coordinates system, this partial differential equation could become a set of uncouple ordinarily differential equation, that set will now be countably infinite; it would not be finite as in discrete multi-degree freedom systems. This type of systems are also known as distributed parameter systems in the sense, if you take any chunk of the beam, it has elastic property; it has inertial property; it dissipates energy unlike in discrete multi-degree freedom systems, where we used the mass and spring and a damper, the mass element stored only kinetic energy; it had only inertial properties, there was no potential energy or stiffness here and so on and so forth. So, they are always known as distributed parameter systems.

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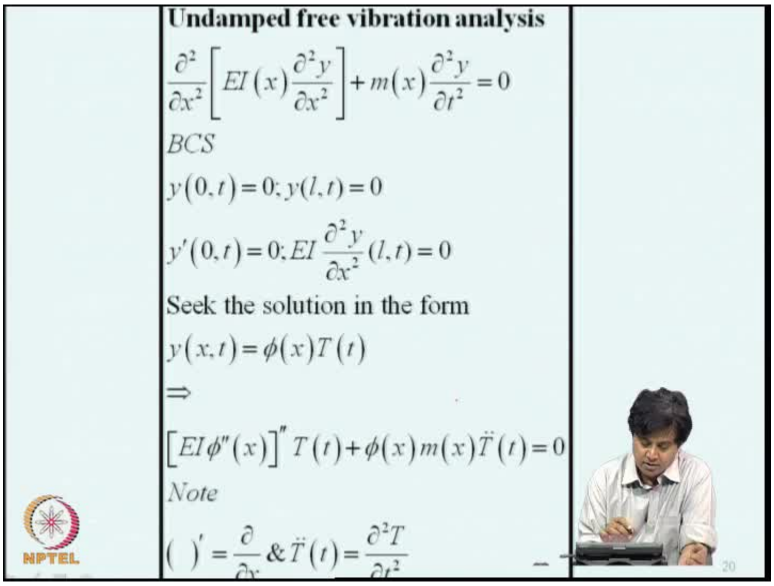

So, as in the case of discrete multi-degree freedom systems, we begin by considering the undamped free vibration analysis. So, this is the equation for undamped free vibration analysis and we seek a solution in the form phi of x into T of t. So, this is the variable separable form that we are trying to see, we are not sure if this is the possible; we have to first establish that such solutions are possible. We substitute into this I get e i phi double prime double prime T of t phi of x m of x T of t is equal to 0; here prime denotes derivative with respect to x and dot denote derivative with respect to time; so t double dot of t is dou square T by dou T square.

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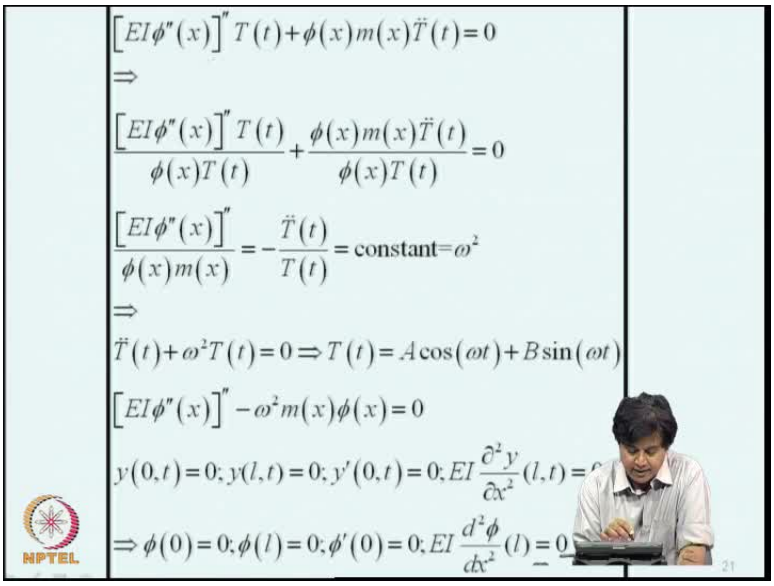

Now, if you divide both sides of this equation phi and T, we get and performs some simplification; we can express this equation as on the this part of the equation, we have function quantity, which are functions of x alone; this is part of the equation, where we have quantity, which are functions of time alone. The variable separable form of this solution requires that these two ratios should be equal.

Now, since this is function of x alone, if I change x, only this quantity can change and this cannot change; similarly, if I change T alone, only this quantity can change only this quantity can change this cannot change. Therefore, if these ratios have to be equal for all x and T, they have to be independently equal to same constant and if we now take that constant as omega square, I get for T of t as solution a cos omega T plus B sin omega T, and for phi of x, I get an equation as displayed here. This phi of x is now that dependent variable, which is now function of only space variable x; so these are all ordinary derivatives. Now, so the boundary conditions on y can now be expressed in terms of boundary conditions on phi y of (0, t) is 0. Therefore, phi of 0 into T of t is 0; if that has to be true for all t, phi of 0 must be 0; similarly phi of l is 0, phi prime of 0 is 0 and bending moment EI d square phi by d x square at l equal to 0.

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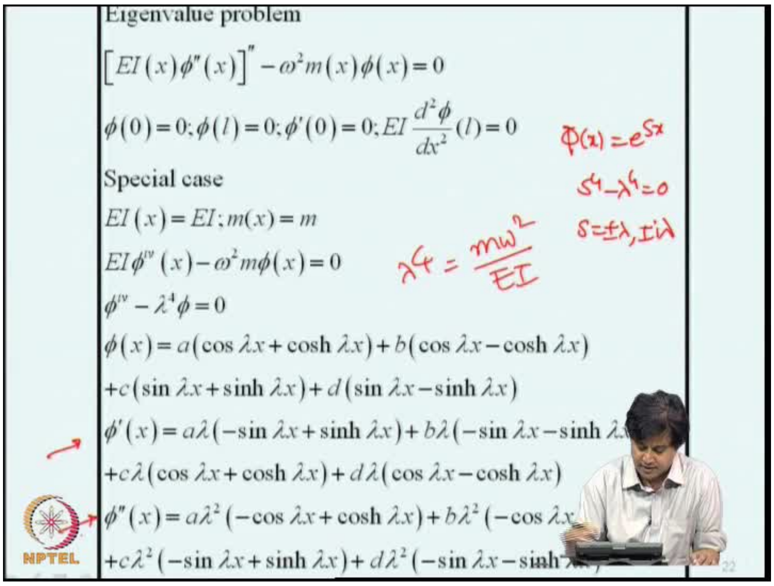

So, we get now a fourth order ordinary differential equation in phi of x along with the associated four boundary conditions; so this is displayed here. You can see here that phi of x is a solution phi of x equal to 0 is a solution, because you substitute phi of x equal to 0, this equation is satisfied and all the boundary conditions are satisfied; this is true for any value of omega. Now, trivial solutions are of no interest to us; so we seek now the to answer this question, can phi of x naught equal to 0 be a solution to this, for some values of omega. So, this is statement of the Eigen value problem; now the operator is not a matrix operator, but differential operator.

So, to discuss this further, to keep the analysis somewhat simple, we will consider E I of x to be E I; m of x to be m, that means, the beam is homogenous and if I now substitute make this simplification here, I can write this E I phi 4 minus omega square M phi equal to 0 and if I divide both sides by E I and denote lambda to the power of 4 as m omega square by E I, I get this equation and this equation can be solved. So, you can assume phi of x to be phi of x can be assumed to be E raise to S x. So, the characteristic equation will be S to the power of 4 minus lambda to the power 4 equal to 0; so we get four roots S is plus minus lambda plus minus I lambda; so associated with plus minus lambda, you have sin h and cos h functions; with I lambda, you have sin lambda x and cos lambda x function. So, for certain reasons, which will become apparent, we will write they now the solution in terms of the function cos lambda x plus cos h lambda x; cos lambda x minus cos h lambda x and similarly, sin lambda x plus sin h lambda x and sin lambda x minus sin h lambda x; a, b, c, d are the four constants that we have to determine by using these four boundary conditions.

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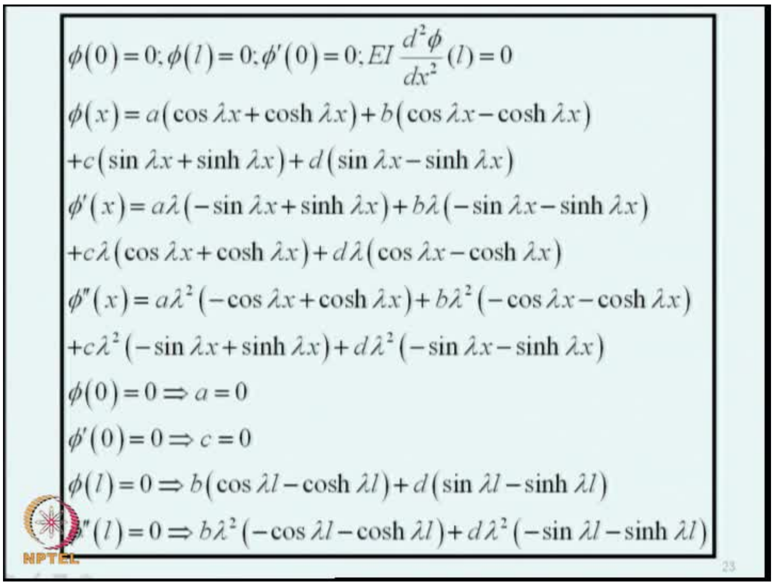

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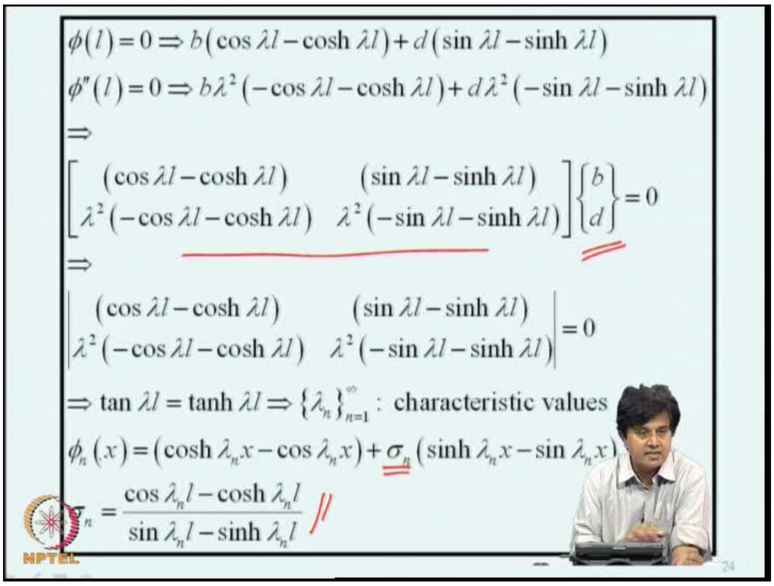

So, to implement the boundary conditions, I need phi prime and phi double prime; so a simple differentiation of phi of x, once, leads to this equation; twice, leads to this equation. Now, imposition of boundary condition; now phi of 0 is 0 and phi of l is 0; so phi of 0 is 0 means, if you now look at phi of x equal to 0, this will be nonzero, but all other terms are 0 at x equal to 0; therefore, I get a equal to 0 that is the particularly the advantage of writing the complementary function in the form that we have done; phi dash of 0 is 0; therefore, I get c equal to 0. So, I am left with now two constants b and d and to determine them, I use the condition phi of l equal to 0 and phi double prime of l equal to 0, and this leads to two equations, which are linearly b and d and this can be put in the matrix form as shown here for b and d here, and for non-trivial solutions of b and d, the determinant of this coefficient matrix must be equal to 0 and that leads to an equation known as characteristic equation. And in this particular case, you have tan lambda l equal to tan h lambda l as the characteristic equation, and if you solve them, you can get infinite set of roots this a transcendental equation therefore, there can be infinite set of solutions and these are known as characteristic values.

Associated with each characteristic value, there will be an Eigen function and that is displayed here and sigma n is the constant that appears here is this. type of the These are the mode shapes for in this case of propped cantilever beam, but similar expressions for various types of single span beam like simply supported beam, free beam, both sides clamp 1, end free other end clamp, etcetera there widely studied there are all catalog in text books.

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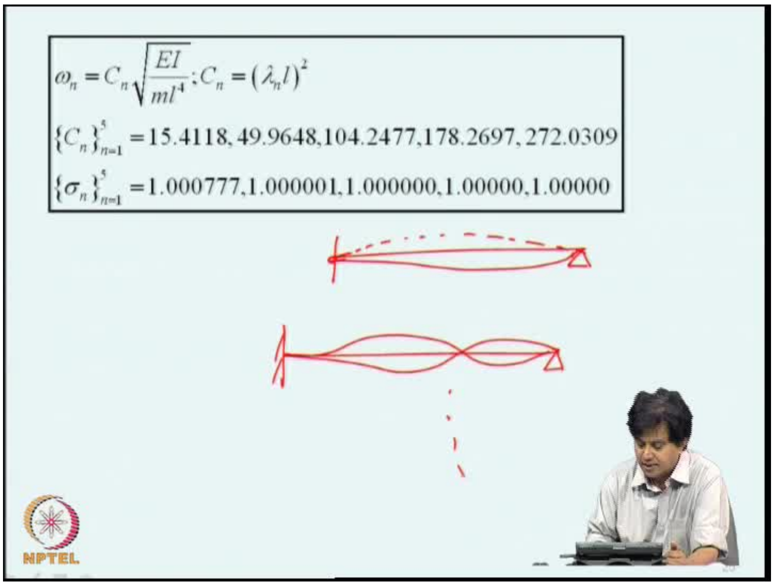

So, for this particular case we get these constants C n and sigma n to be this, based on which we can construct the motions. If the beam is simply supported, the solution is somewhat simpler ; the natural frequencies are given exactly by this and the mode shapes are harmonic in space; I leave this is an exercise for you to prove this. And if you plot the mode shapes for different value of n on x-axis, I have the span of the beam x by l normalized and here, I am plotting the mode shape for n equal to 1, I get this red line; for n equal to 2, the blue line; n equal to 3, the green line and so on.

For the case of propped cantilever, the mode shapes can be sketched here; the first mode will be like this; the slope and the value of the displacement will be 0 that means the beam oscillates in this form in the first mode.

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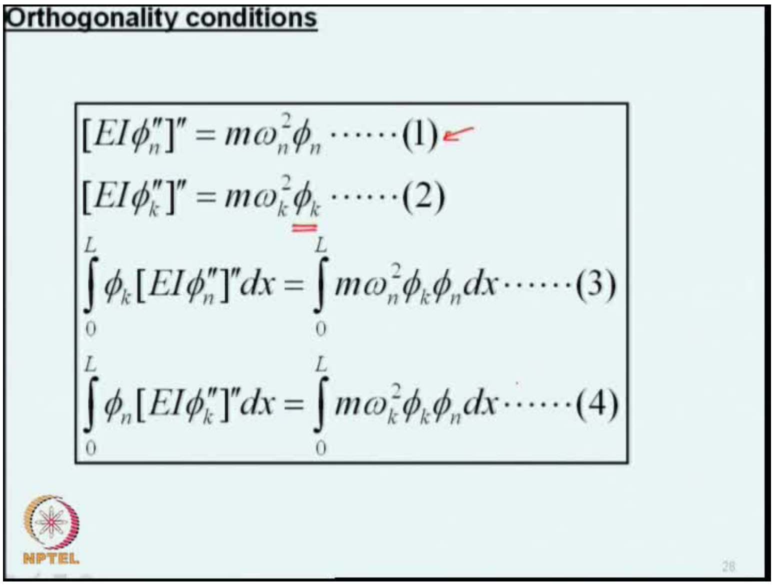

The second mode will be something like this; so you can construct this mode shapes for higher model values also. Now, again taking q from orthogonaility relations of eigen vectors that we encountered in dealing with discrete multi-degree freedom systems, we can now consider the question of do these eigen functions satisfies any orthogonaility relations? To see that, we consider 2 modes n and k and consider the eigen pair omega n and phi n and omega k phi k and as per our formulation, the eigen pair should satisfy these two equations. So, what I do is, I multiply equation 1 by phi k and integrate from 0 to l and multiply equation one by equation 2 by phi n and integrate from 0 to l, so I get this at end of that exercise, I get these pair of equations.

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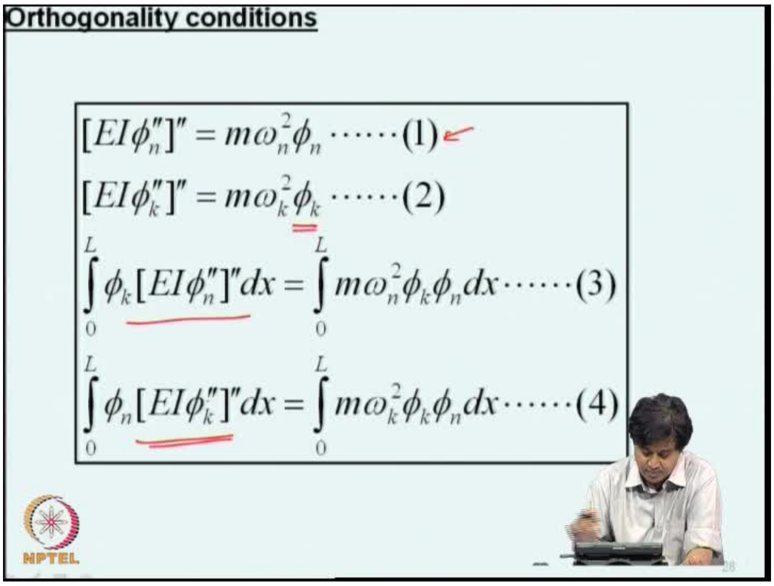

Now, if you now look at the integral that appears on the right hand left hand side, if you do integration by parts, we can integrate once; I get 0 to l E I phi k double prime; double prime becomes E I phi k single prime; phi k remains as it is, then phi n, I have to now differentiate this phi n phi prime E I phi k double prime d x. Now, if I carry out this integration once more, I get this expression; this simplifies to this and then, I get this integer.

Now, if you look at the nature of this integral for typical boundary conditions like a hinged end, a clamped end, free end and sliding end, we can see that for example, if you consider hinged end, the displacement will be 0 and this quantity E I phi k double prime will not be 0; if you raised displacement, you get a reaction and this reaction is a shear force. In a clamped end, the same is true; phi k is 0, there will be a reaction shear force; how about the free end? Displacement is not 0, but this quantity E I phi k double prime is 0, because there is no shear force at the free end.

So, we see that the terms inside these two braces would cancel out, become 0 for any of the boundary condition that where typically interested in modeling single span beams. So, this integral therefore, now simply becomes only this; this become 0; this become 0; I am left with only this. Now, we analyze this similar analysis will lead to a similar simplification for the second integral also.

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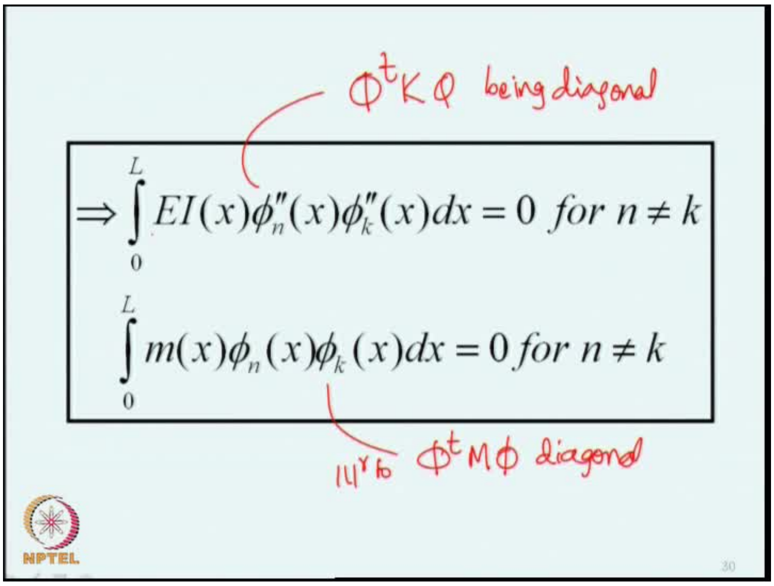

Now, based on that, we can show that phi n double prime of x and phi k double prime of x into E I of x integral 0 to l is 0, for n not equal to k; and m of x phi n of x phi k of x d x equal to 0 for n not equal to k; these are the orthogonaility relations satisfied by the eigen functions. Now, this this second equation is similar to phi transpose m phi being diagonal and this is similar to phi transpose k phi being diagonal for discrete systems. This transposition and addition are now of course replaced by integrations. So, in terms of mathematical content, these two are quite similar.

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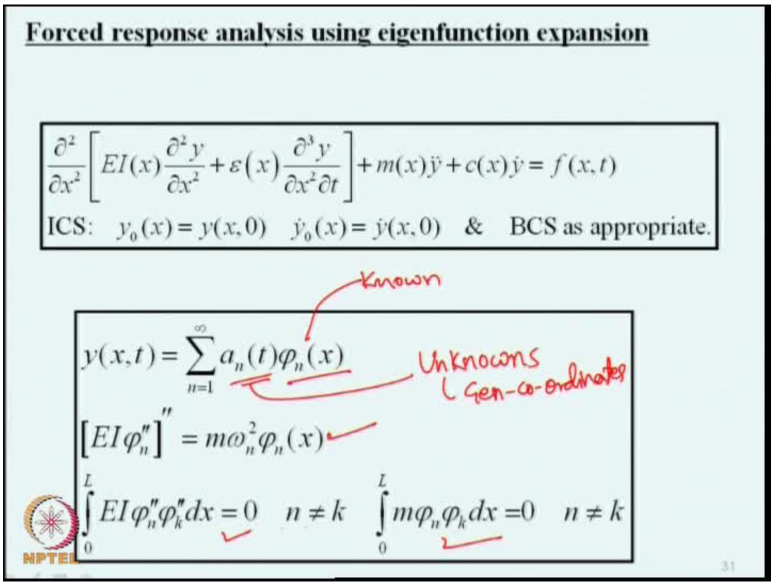

Now, what is the use of finding undamped normal modes and natural frequencies? they should facilitated That should facilitated us to carry out forced response analysis. So, let us consider the problem of forced response analysis under specified initial conditions and specified boundary conditions; this epsilon x is now a constant that appears in modeling strain rate dependent damping and c of x is the functions that appears in modeling velocity dependent damping. We assume y of x comma t as n equal to 1 to infinity a n of t phi n of x; now phi n of x are mean; now at this stage these are known; so these are unknowns and these are the generalized coordinates; this phi n of x are the eigen functions that we just now determined and these satisfy these two orthogonaility relations, with one with respect to E I and one with respect to m.

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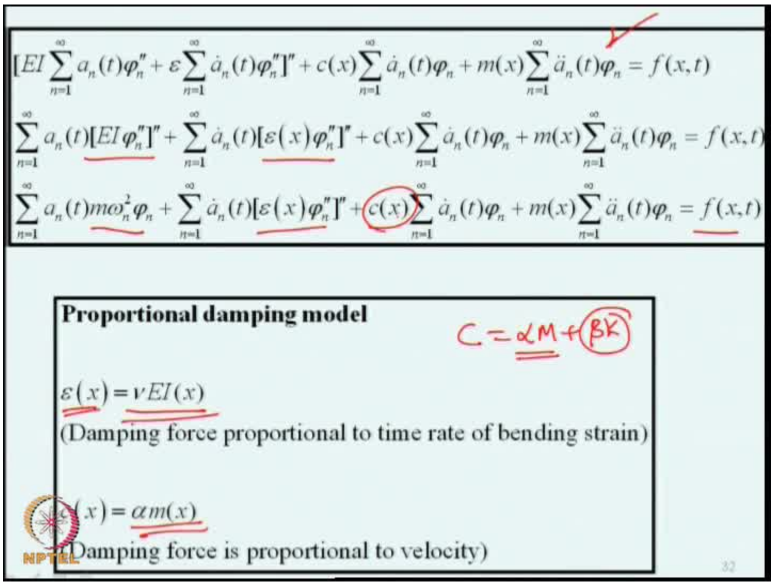

So, E I is the function of x; m is the function of x that is what is implied when we write E I inside this integral and m inside this integral. So, we substitute now this into the governing equation this equation into the governing equation and I get the equation shown here. Now, for E I phi n double prime double prime that appears here, I will write it as m omega n square phi n, because that is what is the definition of phi n of x; this epsilon x into phi n double prime write now I am retaining as it is; c of x is retained as it is; m of x is this and this is f of (x, t) external force.

Now, as we discussed during while discussing multi degree freedom system discrete multi-degree freedom systems, we saw that the c matrix need to be restricted; so the phi transpose if phi is diagonal. So, similarly, here one of the models was c equal to alpha m plus beta k. Now, this part of the model can be easily generalized to beam problem by considering this c of x, which appears here as alpha into m f of x. This part of simplification can be extended generalized to continuous system by assuming that this function epsilon x is some nu into E I of x.

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$$\sum_{n=1}^{\infty} a_n(t) m\omega_n^2 \varphi_n + v\sum_{n=1}^{\infty}\dot{a}_n(t)[EI(x)\varphi_n'']'' + \alpha m(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t) m\omega_n^2 \varphi_n + v\sum_{n=1}^{\infty}\dot{a}_n(t) m\omega_n^2\varphi_n(x) + \alpha m(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t)\omega_n^2\int_0^L m(x)\phi_k\phi_n dx + v\sum_{n=1}^{\infty}\dot{a}_n(t)\omega_n^2\int_0^L m(x)\phi_k\phi_n dx + \sum_{n=1}^{\infty}\dot{a}_n(t)\int_0^L c(x)\phi_k\phi_n dx + \sum_{n=1}^{\infty}\ddot{a}_n(t)\int_0^L m(x)\phi_k\phi_n dx = \int_0^l \phi_k(x) f(x,t)dx$$

So that we will be able to now use orthogonality relations, even for terms involving epsilon of x and c of x how that happens? So, I will now write for epsilon of x and nu into E I of x; therefore, it is here and this itself is now equal to m omega n square phi n, because E I of x phi n double prime is m omega n square phi n and this alpha m x remains as it is. Now, this is the equation that we need to tackle further.

So, what we do? I multiply by phi k of x and integrate from 0 to l; if you now do that, I get these integrals- first term is m phi n is already there, m of x phi k phi n d x plus nu into a n dot omega n square m of x phi k phi n d x; similarly, this c of x is actually alpha into m of x, this remains as it is; this is a n double dot of t m of x phi k phi n and the right hand side, I have 0 to l phi k of x f of (x, t) d into d x.

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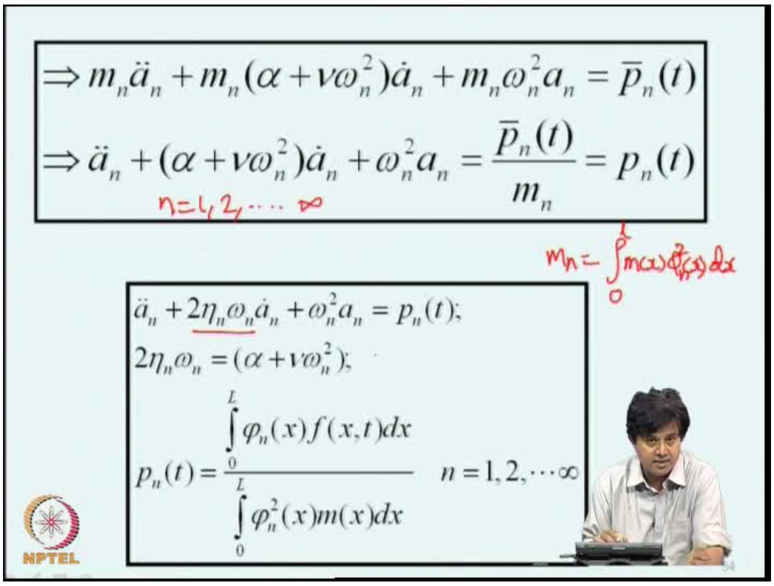

So, this will be a function of time alone and now, I can use orthogonaility relations with respect to m, this is equal to 0, for k not equal to n; this is also 0, for k not equal to n; this is 0, for k not equal to n; similarly, this is 0, for k not equal to n. So, only the case of k equal to n would remain and I get the equation m n a n double dot plus some m n alpha plus nu omega n square a n dot plus m n omega n square a n is p n of t; so where m n I have not mass normalized; so we will have to use this m of x phi n square of x d x. If the Eigen functions are mass normalized, m n will be equal to 1; write now we have not done that, so we are just leaving it as it is.

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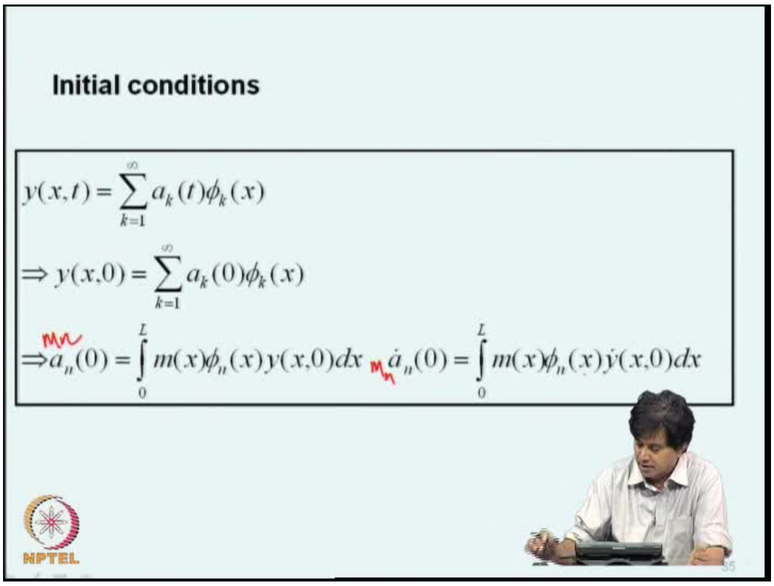

So, for the generalized coordinates, I get this equation n equal to 1, 2 to infinity. So, this can be cost in the standard form by writing alpha plus nu omega n square as 2 eta n omega n. So, this is the family of single degree freedoms systems with n running from 1 to infinity and we know how to tackle them. We still need to find out the initial conditions to be applied to a n of t; so to do that, I have this y of (x, t) a k of t phi k of x; so if you evaluate this at t equal to 0, I get a k of 0 phi k of x.

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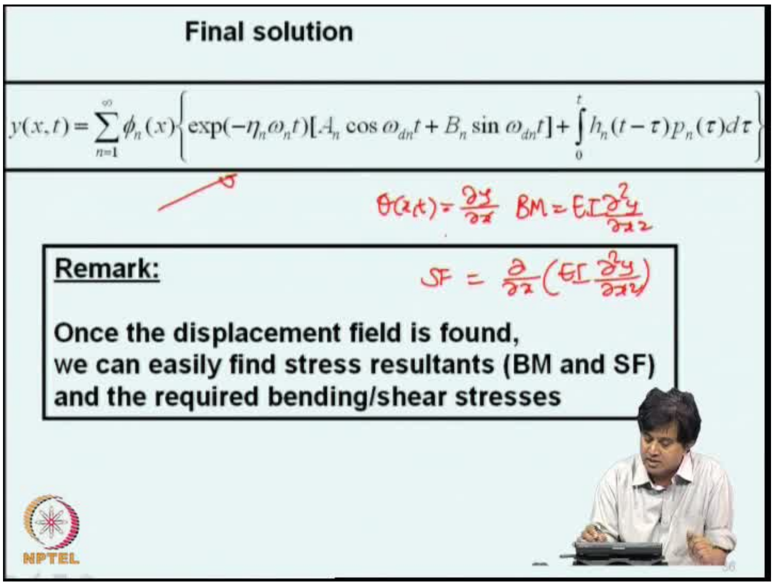

Now, I multiply both sides by m of x into phi n of x and integrate from 0 to l; so on the right hand side, I get m of x phi n of x y of (x, 0); on the right hand side, I get a n of 0 into m n; this m n should be here; this must equal to 0. So, a n of 0 is determined; a n dot of 0 is determined and we are now into business, because we have now uncoupled the equation; we have find out the initial conditions; so I can integrate this. We know now, what we know for single degree freedom system has to be applied for a long number of times and we get that solution and this is the final solution. This is the displacement field; moment I know the displacement field, I can differentiate this; you want theta, that is, the slope is dou y by dou x; you want bending moment, that is, E I dou square y by dou x square; you want shear force, it is dou by dou x of E I dou square y by dou x square. So, once y is known, I can find out all this and once you know bending moment, you can compute the bending stresses; if you know shear force, you can find out the shear stresses; if you want principal stresses, you can do the analysis of state of stress at any point of the beam; find out principal stresses, principal directions and you can check if failure criteria like avon mises failure or maximum principal stress failure criteria, etcetera or you know violated or not. So, you can do any engineering analysis that is of interest to you.

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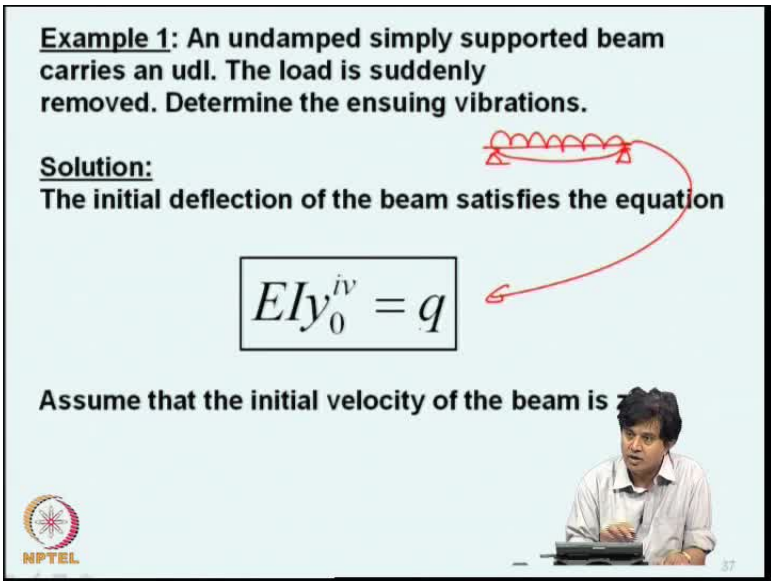

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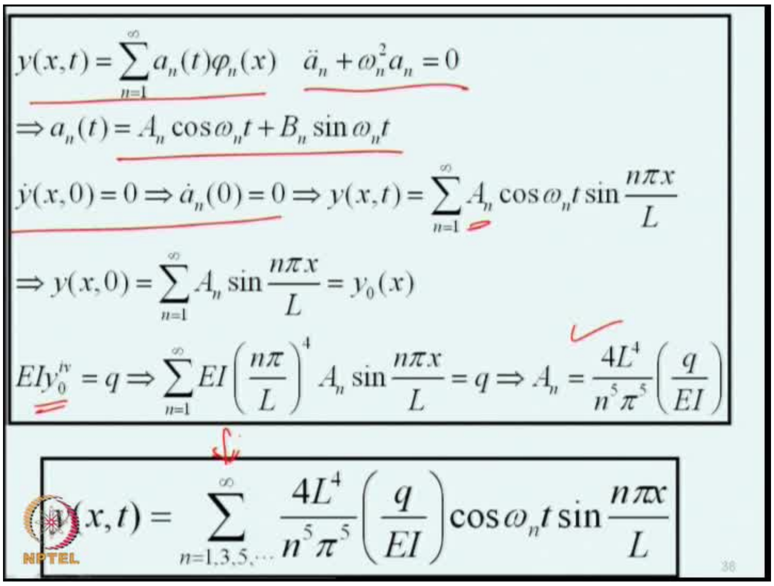

Now, I will consider a few simple problems; suppose we consider undamped simply supported beam carry carrying a u d l, suppose the load is suddenly removed, how does the structure vibrate? So that means, I have a beam, which is carrying some u d l and it has deformed now and this is the equilibrium equation for that and now, I suddenly remove the load; so this structure would has to oscillate and that oscillation is essentially due to this prescribed initial conditions. The initial displacement profile is due to the applied load q and for t positive that load does not exist; therefore, the structure oscillates and we assume that t equal to 0, the beam velocity is 0. So, we can we can assume the model solution as it is; these are the equations. Now, we are considering undamped free vibration.

So, a n of t is given by this and initial conditions we have to utilize to find out this a n and b n initial velocity is 0; therefore, a n dot immediately goes to 0; so b n becomes 0 and I am left with only a n and these a ns can be found out by, you know substituting this solution into the governing equation for y naught and with a little manipulation, you can find out all the a ns and y of (x, t) is given as here. A simple problem you try it out, it will teach you some simple things very fast.

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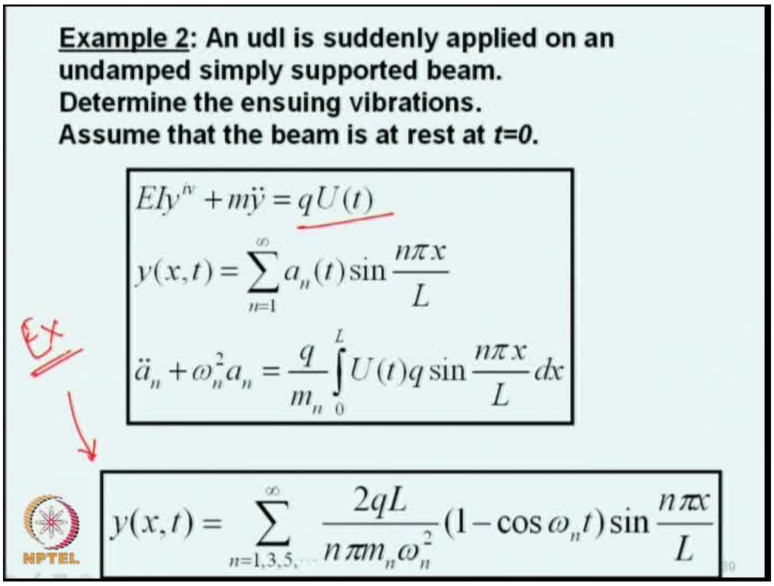

Another problem somewhat similar surrounding; now, a u d l is suddenly applied on an undamped simply supported beam, determine the ensuing vibrations. Assume that the beam is at rest at t equal to 0, that means, some imagine some say cement bag or something falls on a beam and beam vibrates; so it is a suddenly applied load, we use u site step function q u of t and we initially the system is at rest. So, I have given a few steps here, you can run through this and show that y of (x, t) is indeed given by this. So, this is an exercise that you have to complete.

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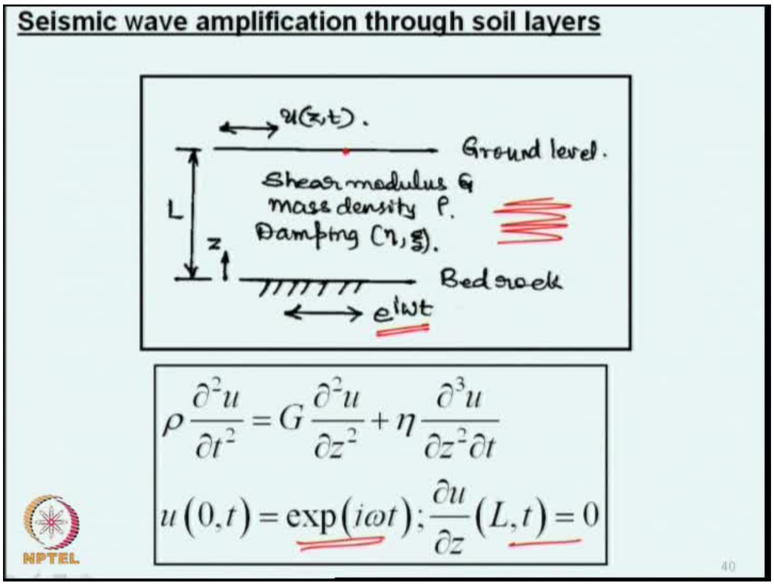

I will briefly touch upon another strategy to solve continuous systems. So, to do that, we will consider a soil layer, suppose this is a bedrock and this is soil layer characterized by shear modulus, mass density and some damping parameters and this is the ground level; suppose at the bed rock level, I apply a unit harmonic displacement and I would like to know what is the amplitude of displacement at the ground level. right So, this ratio is known as amplification factor and if we are close to resonance, this amplification factor can be high and this is of a fundamental interest in earthquake engineering problems to allow for local site conditions, so we need to perform the wave amplification studies.

So, this is a simple model; suppose you assume that this soil layer behaves as a shear beam; you recall when we discussed Kanai Tajimi power spectral density function model, I replace this soil by a single degree freedom system; now I am refining that model and I am replacing this soil layer by shear beam, that means, the beam that oscillates in this manner; u is the displacement and this is the equilibrium equation and I have assuming damping in a certain form. With boundary conditions now, on displacement as shown here; the top edge is the free from applied tractions; therefore, the strain should be 0; so this is the boundary condition at the ground level; this is at the bedrock level.

How do we solve this problem? We cannot directly use the model decomposition method here, because to apply model decomposition method, we need boundary conditions which are not functions of time. this is this is the boundary condition the Here boundary conditions is a function of time therefore, we cannot use the eigen function expansion directly. There is a way to introduce a transformation of dependent variables to make the boundary conditions independent of time in the transform domain and then, subsequently apply normal mode expansion method, that I will considering in the next class.

But now, we will consider a solution to this problem in a using a slightly alternative approach. This is a linear system and it is receiving input that is harmonic; so any response quantity is going to be harmonic in steady state at the driving frequency, that is one of the important properties of linear systems.

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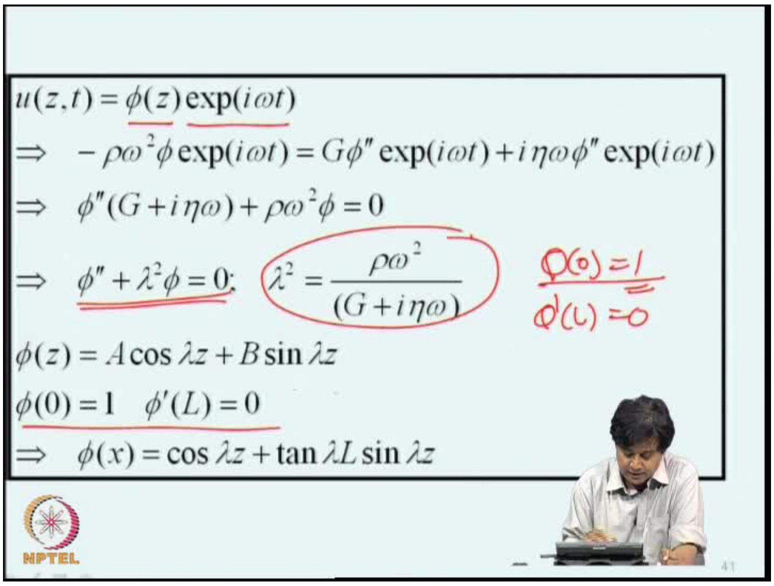

So, consequently what I assume is u of (z, t) is some phi of z into exponential i omega t; this phi of z is the amplification factor dependent on z, it is the space and frequency dependent amplification factor. Now, I can substitute the assume solution, phi of z is unknown now, we substitute here, I get this equation phi double prime g plus i eta omega rho omega square phi equal to 0 and if I now introduce notation lambda square is rho omega square divided by this, I get an equation here; mind you the initial the boundary conditions here are phi of 0 is 1; phi prime l is 0.

So, this is not an Eigen value problem; omega is a given driving frequency and the boundary condition here is non-homogeneous. So, this is not a eigen value problem, it is simply you have to found out the response; response phi equal to 0 is not solution to this, because phi equal to 0 will not satisfy this boundary condition; so this is not an Eigen value problem, although it has some resemblances to the what we discussed earlier.

So, we put phi of z is a cos lambda z plus sin lambda z. We have a phi of 0 phi of 0 is 1 and phi prime l equal to 0 based on that, I get phi of x has cos lambda z tan lambda l sin lambda z.

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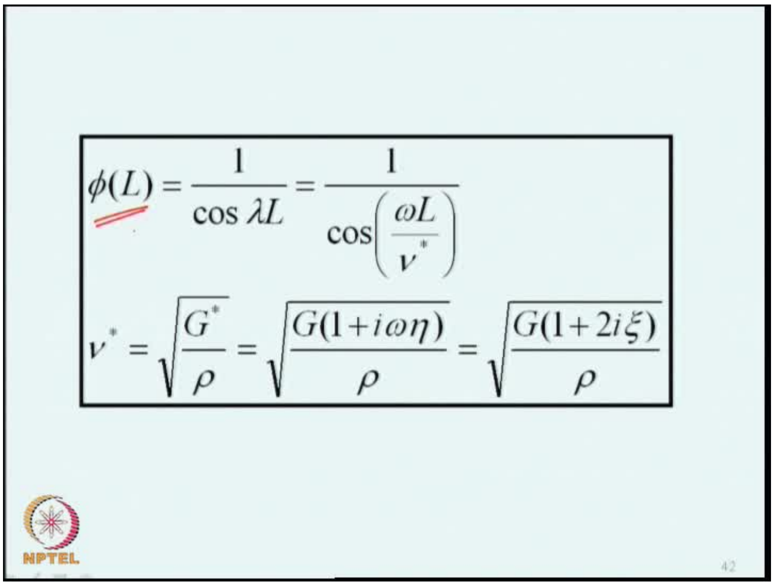

Now, if you want now solution at x equal to l that is at the ground level, I get 1 by cos lambda l and if I now substitute for lambda l, I get this and where this nu star is the shear wave velocity complex valued shear wave velocity, because it includes effect of damping under the driving frequency; we have got the amplification factor. Because at the bedrock level, the applied amplitude is unity and phi of l therefore directly represents the amplification factor.

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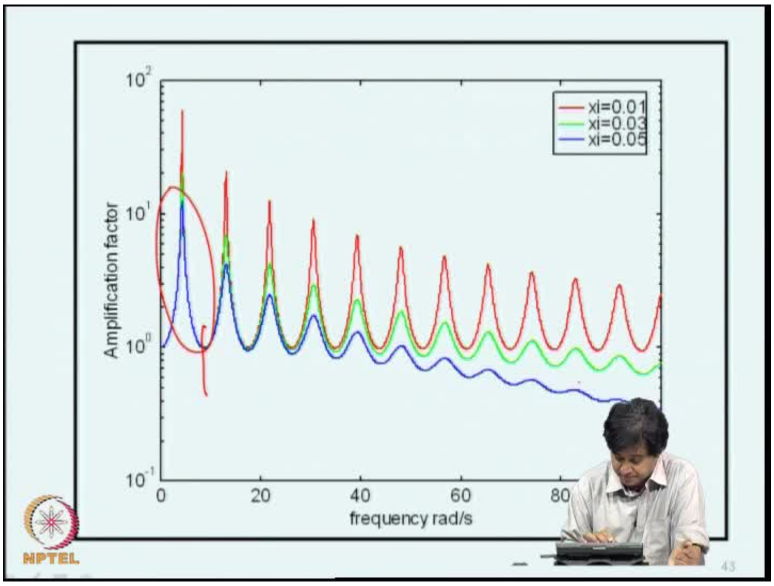

If you plot phi of l as a function of frequency, the amplitude of phi of l you see that, it has this characteristic behavior, where there are several peaks, which are uniformly spaced and these are the nothing but the points, where these peaks occur are nothing but the natural frequencies of the soil layer. In Kanai Tajimi power spectral density model, we included only this part; we had a single degree freedom model; so this can be viewed as a as a preparatory step towards developing a Kanai Tajimi power spectral density model, where soil layer is taken as continuous system than as a single degree freedom system.

We will stop this lecture here and we will continue with discussion on beams in the next lecture.