

Soil Structure Interaction
Prof. Kousik Deb
Department of Civil Engineering
Indian Institute of Technology - Kharagpur

Lecture 38
Beams on Elastic Foundation (Contd.,)

In my previous lecture I was discussing about a beam resting on two parameter soil medium and derived the basic differential equation. Then for infinite beam I determined the constants. Today I will discuss how to determine the settlement and bending moment for an infinite beam resting on two parameter soil medium. And then I will discuss about the beam with finite length resting on two parameters soil medium.

(Refer Slide Time: 01:01)

Beams on Two Parameter Soil Medium

Beam: (i) Infinite Beam (Application: The Railroad Tracks, long strip footings, combined footings): With Finite Width
(ii) Semi-Infinite Beam: With Finite Width
(iii) Beam with Finite Length (Continuous strip footings, combined foundations): With Finite Width and Under Plane-strain condition

$EI \frac{d^4w}{dx^4} = -p + q$
 $EI \frac{d^4w}{dx^4} + b^4w = b^4q$
 $EI \frac{d^4w}{dx^4} - b^4w = b^4q$
 $b = \sqrt[4]{\frac{K}{4EI}}$
 $K = \frac{K_u - G_H}{b}$
 $q = K_u w - G_H \frac{dw}{dx}$

$b = \text{width of the beam}$
 $q = \text{surface load}$
 $K = \text{spring constant}$
 $K_u = \text{unit spring constant}$
 $G_H = \text{shear modulus}$

WALL OR STRIP FOOTING

$b = \text{width}$

<https://www.indiamart.com/prodetail/single-railway-track-1560469348.html>
<https://medium.com/@stulpaire1/10-types-of-foundations-used-in-constructi-2b0cf1e0e0e0>

NPTEL Online Certification Course
IIT Kharagpur

This was the problem that was being discussed in the last class where a beam is resting on two parameter soil medium and the basic differential equation was also given.

(Refer Slide Time: 01:13)

$at\ x=0 \quad \frac{dw}{dx} = 0 \quad \beta C_4 - \alpha C_3 = 0 \quad \boxed{C_4 = \frac{\alpha}{\beta} C_3}$
 $\frac{dw}{dx} = -\lambda e^{-\lambda x} \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 \sin \lambda \beta x - \lambda \alpha e^{-\lambda x} (-\alpha \sin \lambda \beta x + \beta \cos \lambda \beta x)$
 $Min \quad \frac{dw}{dx} = -\lambda \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 e^{-\lambda x} \left[(\beta^2 - \alpha^2) \sin \lambda \beta x + 2\alpha \beta \cos \lambda \beta x \right]$
 $\frac{d^2w}{dx^2} = \lambda^3 \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 e^{-\lambda x} \left[(\beta^2 - \alpha^2) \sin \lambda \beta x + 2\alpha \beta \cos \lambda \beta x \right]$
 Shear force $Q = -EI \frac{d^3w}{dx^3} + b^2 G H \frac{dw}{dx} = -\frac{P}{2}$
 $Q|_{x=0} = -\frac{P}{2}$
 Shear force (H) $= b^2 G H \frac{dw}{dx} - EI \frac{d^3w}{dx^3} = -\frac{P}{2}$
 or $EI \frac{d^3w}{dx^3} = \frac{P}{2}$
 $\boxed{C_3 = \frac{P}{4EI \lambda^3 \alpha (\beta^2 + \alpha^2)}} \quad x=0 \quad -EI \frac{d^3w}{dx^3} = -\frac{P}{2}$

The value of the constants was determined where $C_1 = C_2 = 0$, $C_3 = \frac{P}{4EI \lambda^3} \frac{1}{\alpha(\beta^2 + \alpha^2)}$ and the

relation between C_3 & C_4 was given by: $C_4 = \frac{\alpha}{\beta} C_3$.

(Refer Slide Time: 02:08)

$EI \frac{d^4w}{dx^4} - b^2 G H \frac{d^2w}{dx^2} + b^2 K w = b^2 q$
 In case of Plane-strain Problem $b^2 = b$ where $b = \text{width of the beam}$
 E^* should be replaced by $\frac{E}{1-\mu_p}$ where $\mu_p = \text{Poisson's Ratio of the beam}$
 In case of beam with finite width $b^2 = b \left[1 + \frac{G H}{\sqrt{b^2 K}} \right]$ b is the width of the beam
 $K = \text{Krylov's } \mu$
 $N_{xx} = 0 \quad E^* = E$
 $EI \frac{d^4w}{dx^4} - b^2 G H \frac{d^2w}{dx^2} + b^2 K w = 0$
 $w = e^{-\lambda x} (C_1 \cos \lambda \beta x + C_2 \sin \lambda \beta x) + e^{-\lambda x} (C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x)$
 $\lambda = \sqrt[4]{\frac{b^2 K}{4EI}}, \quad \alpha = \sqrt{1 + \frac{G H}{K} \lambda^2}, \quad \beta = \sqrt{1 - \frac{G H}{K} \lambda^2}$
 Infinite beam $w \rightarrow 0$ if $x \rightarrow \infty \quad C_1 = C_2 = 0$
 $w = e^{-\lambda x} (C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x)$
 $\frac{dw}{dx} = \lambda e^{-\lambda x} [(\beta C_4 - \alpha C_3) \cos \lambda \beta x + (-\beta C_3 - \alpha C_4) \sin \lambda \beta x]$

The strip footing was considered as a beam to analyse and as the length of the strip footing is infinite, it was considered as the width for the beam. The width of a strip footing is generally finite and it is considered as the length of the beam. These two considerations enabled to use the case where a beam is of finite length under plane strain condition (due to the infinite width).

(Refer Slide Time: 03:31)

$w = \frac{P}{4EI\lambda^3} \frac{e^{-\lambda x}}{\alpha(\beta^2 + \alpha^2)} \left(\cos \lambda \beta x + \frac{\alpha}{\beta} \sin \lambda \beta x \right)$
 $M = \frac{P}{4\lambda\alpha\beta} e^{-\lambda x} \left(-\alpha \sin \lambda \beta x + \beta \cos \lambda \beta x \right)$
 $\lambda = 4 \sqrt{\frac{k b^3}{4EI}}$, $\alpha = \sqrt{1 + \frac{GH}{k} \lambda^2}$, $\beta = \sqrt{1 - \frac{GH}{k} \lambda^2}$

Example
 Beam on Two-Parameters
 S.T. Medium
 $P = 100 \text{ kN}$
 $b = 0.25 \text{ m}$
 $h = 0.3 \text{ m}$
 $G = 20000 \text{ kN/m}^2$
 $E = 10^7 \text{ kN/m}^2$
 $I = \frac{1}{12} (0.25)^3 (0.3)^3 = 1.67 \times 10^{-4} \text{ m}^4$
 $EI = 1.67 \times 10^3 \text{ kN-m}^2$

Infinite Beam
 $b^* = b \left[1 + \sqrt{\frac{GH}{b^2 k}} \right] = 0.25 \left[1 + \sqrt{\frac{20000 \times 0.3}{(0.25)^2 \times 10000}} \right] = 1.025 \text{ m}$
 $\lambda = 4 \sqrt{\frac{b^* k}{4EI}} = 4 \sqrt{\frac{1.025 \times 10000}{4 \times 1.67 \times 10^3}} = 1.11 \text{ m}^{-1}$
 $\alpha = \sqrt{1 + \frac{GH}{k} \lambda^2} = \sqrt{1 + \frac{20000 \times 0.3}{10000} (1.11)^2}$

If the values of C_3 & C_4 are substituted in the deflection expression:

$$w = \frac{P}{4EI\lambda^3} \frac{e^{-\lambda x}}{\alpha(\beta^2 + \alpha^2)} \left(\cos \lambda \beta x + \frac{\alpha}{\beta} \sin \lambda \beta x \right)$$

The bending moment expression will be:

$$M = \frac{P}{4\lambda\alpha\beta} \left(-\alpha \sin \lambda \beta x + \beta \cos \lambda \beta x \right)$$

$$\text{where, } \lambda = 4 \sqrt{\frac{b^* k}{4EI}}, \alpha = \sqrt{1 + \frac{GH}{k} \lambda^2}, \beta = \sqrt{1 - \frac{GH}{k} \lambda^2}$$

Now let us solve an example problem of determining the deflection of an infinite beam subjected to concentrated load, $P = 100 \text{ kN}$. The point where the point load is acting is treated as the $(0,0)$ point. The beam cross section is given by $b = 0.25 \text{ m}$ and $h = 1.2 \text{ m}$. The shear modulus value is $G = 20000 \text{ kN/m}^2$, elastic modulus, $E = 10^7 \text{ kN/m}^2$, moment of inertia, $I = 1.67 \times 10^{-4} \text{ m}^4$, $EI = 1.67 \times 10^3 \text{ kN-m}^2$ and $k = 10000 \text{ kN/m}^2$.

Here the beam given is infinite as it has a finite width, 0.25 m . In general, infinite beam would have finite width. If the beam is under plane strain condition, the b value will be chosen as unit. But, here a finite value for beam width is given and so the b^* value should be determined.

$$b^* = b \left[1 + \sqrt{\frac{GH}{b^2 k}} \right]$$

$$b^* = 0.25 \left[1 + \sqrt{\frac{20000 \times 0.3}{0.25^2 \times 10000}} \right] = 1.025 \text{ m}$$

Similarly, the other constants will be found out:

$$\lambda = \sqrt[4]{\frac{b^*k}{4EI}} \Rightarrow \lambda = \sqrt[4]{\frac{1.025 \times 10000}{4 \times 1.67 \times 10^3}} = 1.11 \text{ m}^{-1}$$

$$\alpha = \sqrt{1 + \frac{GH}{k} \lambda^2} \Rightarrow \alpha = \sqrt{1 + \frac{2000 \times 0.3}{10000} \times 1.11^2} = 1.32$$

(Refer Slide Time: 12:14)

$$\beta = \sqrt{1 - \frac{GH}{k} \lambda^2} \Rightarrow \beta = \sqrt{1 - \frac{20000 \times 0.3}{10000} \times 1.11^2} = 0.51$$

Now, the deflection can be determined by the expression:

$$w = \frac{P}{4EI\lambda^3} \frac{e^{-\lambda x}}{\alpha(\beta^2 + \alpha^2)} \left(\cos \lambda \beta x + \frac{\alpha}{\beta} \sin \lambda \beta x \right)$$

If the deflection below the point load is to be found out, the x value in the above expression should be replaced with 0 and so the expression reduces to:

$$w|_{x=0} = \frac{P}{4EI\lambda^3} \frac{1}{\alpha(\beta^2 + \alpha^2)}$$

$$\Rightarrow w|_{x=0} = \frac{100}{4 \times 1.67 \times 10^3 \times 1.11^3} \times \frac{1}{1.32(1.32^2 + 0.51^2)} = 4.14 \text{ mm}$$

Now if the same problem is being solved for the case where beam is resting on Winkler springs, the expression for w at x = 0 will be:

$$w = \frac{P\lambda}{2k}$$

The λ value for the Winkler model is different from here:

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{0.25 \times 10000}{4 \times 1.67 \times 10^3}} = 0.78 \text{ m}^{-1}$$

$$\Rightarrow w|_{x=0} = \frac{100 \times 0.78}{2 \times 0.25 \times 10000} = 15.6 \text{ mm}$$

The k value is not substituted directly, as the k value in the expression is of kN/m^2 and it should be multiplied by only b, but not b^* . In case of Winkler model, $b^* = b$ as there is no shear layer.

For the same beam and loading conditions, the deflection when the beam rests on two parameter medium is 4.14 mm and on Winkler model is 15.6 mm.

(Refer Slide Time: 18:38)

Beam with finite length Resting on Two Parameter Soil medium

Diagram: A beam of length l with a point load P at the center ($x=0$). The beam is supported by two parameter soil medium.

Differential equation: $w = e^{-\lambda x} (C_1 \cos \lambda \beta x + C_2 \sin \lambda \beta x) + e^{\lambda x} (C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x)$

Boundary Conditions are:

at $x=0$, $\frac{dw}{dx} = 0$ i.e. $\beta (C_2 + C_4) = \alpha (C_1 - C_3)$

at $x=0$ Shear force

$Q|_{x=0} = -b^* G H \frac{dw}{dx} = -\frac{P}{2}$

$-EI \frac{d^2w}{dx^2} = -b^* G H \frac{dw}{dx} = -\frac{P}{2}$

$-EI \frac{d^2w}{dx^2} + b^* G H \frac{dw}{dx} = -\frac{P}{2}$

$\frac{P}{2EI\lambda^3} = \alpha_1 (C_3 - C_1) - \alpha_2 (C_2 + C_4)$

Handwritten notes on the left:

$\alpha_1 = (\alpha^2 \beta)$
 $\alpha_2 = 2\alpha \beta$
 $\alpha_3 = \beta^2 - 3\alpha^2 \beta$
 $\alpha_4 = \alpha^2 - 3\alpha \beta$

NPTEL Online Certification Courses
IIT Kharagpur

The next case is where a beam with finite length is resting on a two parameter soil medium. The basic differential equation already derived was:

$$EI \frac{d^4 w}{dx^4} - b^* G H \frac{d^2 w}{dx^2} + b^* k w = b q$$

Consider a beam of length l subjected to a point load, P at $(0,0)$ point and from this point both the ends of the beam are at a distance of $l/2$. The deflection expression for this will be:

$$w = e^{-\lambda x} (C_1 \cos \lambda \beta x + C_2 \sin \lambda \beta x) + e^{\lambda x} (C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x)$$

In the infinite beam case, there were only two constants to determine as two constants were zeros. But here, for a finite beam all the four constants will have to be determined. Also, in the previous case the first term was $e^{\lambda x}$ and the second term was $e^{-\lambda x}$. But here it is changed a bit according to the solution about to be adopted.

For the given loading conditions, at $x = 0$, the slope (dw/dx) should also be 0 and the shear force should be half the load ($-P/2$).

$$\text{At } x = 0: \frac{dw}{dx} = 0 \Rightarrow \beta(C_2 + C_4) = \alpha(C_1 - C_3)$$

$$\text{At } x = 0, \text{ Shear force: } Q|_{x=0} = -b^*GH \frac{dw}{dx} = -\frac{P}{2}$$

If the beam is resting on Winkler springs, the expression can be directly written as $-EI(d^3w/dx^3) = -P/2$. But, here the shear layer contribution should also be considered:

$$\Rightarrow -E^*I \frac{d^3w}{dx^3} = -b^*GH \frac{dw}{dx} = -\frac{P}{2}$$

$$\Rightarrow -E^*I \frac{d^3w}{dx^3} + b^*GH \frac{dw}{dx} = -\frac{P}{2}$$

But, here we already have a boundary condition that the slope is 0 at $x = 0$. So the dw/dx term in the above expression will be 0 and by substituting this value, another condition can be obtained:

$$\frac{P}{2EI\lambda^3} = \alpha_4(C_3 - C_1) - \alpha_3(C_2 + C_4)$$

$$\text{where, } \alpha_1 = (\alpha^2 - \beta^2)$$

$$\alpha_2 = 2\alpha\beta$$

$$\alpha_3 = \beta^3 - 3\beta\alpha^2$$

$$\alpha_4 = \alpha^3 - 3\alpha\beta^2$$

So from the first boundary condition, a relation within the constants is established. Now, four equations are needed to solve these four unknowns. Till now, two boundary conditions are applied and two relations are established.

In the next class, I will apply two more boundary conditions and will get 2 more relationships. If I solve these equations I will get the 4 equations and then the 4 unknowns. Now if I put these 4 unknowns in the basic equation, I will get the deflection, bending moment, shear force and slope for the beam at any point. So, those expressions I will develop in the next class. Thank you.