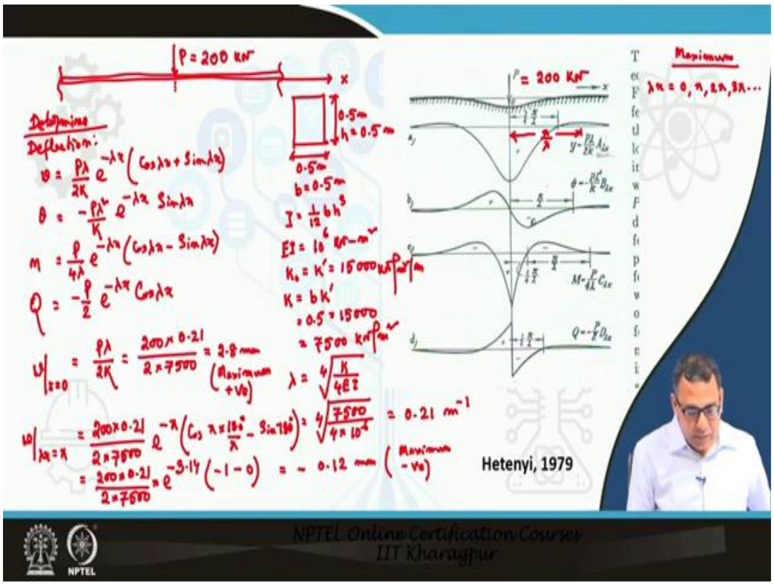
Soil Structure Interaction
Prof. Kousik Deb
Department of Civil Engineering
Indian Institute of Technology- Kharagpur

Lecture 21
Beams on Elastic Foundation (Contd.,)

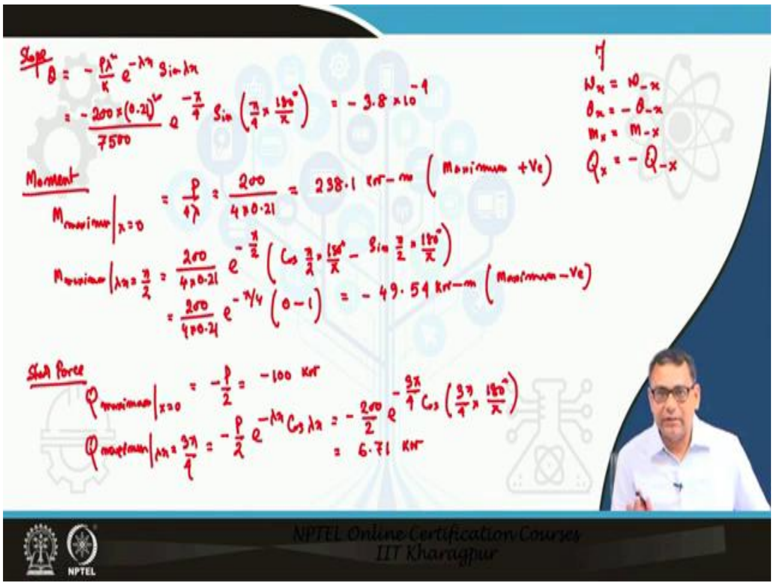
In the last class, I was solving one particular problem of an infinite beam resting on soil subjected to a concentrated load of 200 kN. I already determined the maximum positive and negative deflection under that loading condition. Now let us solve for the slope, bending moment and shear force for the same problem.

(Refer Slide Time: 00:48)



The maximum positive deflection of the beam was found to be 2.8 mm which occurs at the centre (below the point load). The maximum negative deflection which occurs at a distance of π units from the load was found to be -0.12 mm.

(Refer Slide Time: 01:15)



The formula to calculate the slope is:

$$\theta = -\frac{P\lambda^2}{k} e^{-\lambda x} \sin \lambda x$$

For the θ value to be maximum, its derivative (dθ/dx) should be zero which means that (cosλx-sinλx) should be zero. So:

$$\cos\lambda x - \sin\lambda x = 0 \Rightarrow \cos\lambda x = \sin\lambda x \Rightarrow \tan\lambda x = 1$$

So, tanλx should be equal to one for the slope (θ) to be maximum. This means that the slope will be maximum when $\lambda x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$..... Similarly, the bending moment would be maximum if dM/dx is 0 which means cosλx should be 0 (dM/dx is nothing but the shear force and the bending moment can be maximum only if the shear force is 0 at that point). If cosλx is 0, λx can have the values of: $\lambda x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$..... In the same way, for shear force to be maximum, dQ/dx should be 0 which implies that cosλx+sinλx should be 0 meaning that tanλx should equal -1.

$$\cos\lambda x + \sin\lambda x = 0 \Rightarrow \cos\lambda x = -\sin\lambda x \Rightarrow \tan\lambda x = -1$$

To satisfy this condition, λx can have the values of: $\lambda x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$....

The points of maximum and minimum values of all the four quantities are shown in the slide and their distances from the point of application of load are derived above.

The various values of λx for various quantities to be maximum (*under concentrated load acting on an infinite beam*) are summarised below:

For deflection to be maximum: λx = 0, π, 2π.....

For slope, θ to be maximum: $\lambda x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots$

For bending moment, M to be maximum: $\lambda x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

For shear force, Q to be maximum: $\lambda x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4} \ldots$

Let us apply these values in the example problem. Firstly, the slope will be maximum at λx = π/4 which should be substituted in the formula for slope:

$$\theta = -\frac{P\lambda^2}{k} e^{-\lambda x} \sin \lambda x$$

$$\Rightarrow \theta = \frac{200 \times (0.21)^2}{7500} e^{-\frac{\pi}{4}} \times \sin\left(\frac{\pi}{4} \times \frac{180}{\pi}\right) = -3.8 \times 10^{-4}$$

This is the value of the slope at a distance of π/4 from the point of application of load which is the maximum value for this loading condition.

Next, let us calculate the maximum bending moment which occurs at λx = 0. When λx is 0, the formula for bending moment will be reduced to:

$$M_{\max}\Big|_{\lambda x=0} = \frac{P}{4\lambda} \Rightarrow M = \frac{200}{4 \times 0.21} \Rightarrow M = 238.1 kN - m \text{ (maximum positive)}$$

This is the value of the maximum positive bending moment in the beam for the given point load.

The bending moment would be critical to study at another point, where it turns maximum negative in the beam. This occurs at a distance of π/2λ from the point of application of load. So, by substituting λx=π/2 in the following formula of bending moment, we can determine the maximum negative bending moment in the beam:

$$M = \frac{P}{4\lambda} e^{-\lambda x} (\cos \lambda x - \sin \lambda x)$$

$$\Rightarrow M_{\max}\Big|_{\lambda x=\frac{\pi}{2}} = \frac{200}{4 \times 0.21} \times e^{-\frac{\pi}{2}} \times \left[\cos\left(\frac{\pi}{2} \times \frac{180}{\pi}\right) - \sin\left(\frac{\pi}{2} \times \frac{180}{\pi}\right)\right]$$

$$\Rightarrow M_{\max}\Big|_{\lambda x=\frac{\pi}{2}} = \frac{200}{4 \times 0.21} \times e^{-\frac{\pi}{2}} \times (0-1) = 49.54 kN - m \text{ (maximum negative)}$$

This is the value of the maximum negative bending moment in the beam for the given point load.

Similarly shear force will be determined using the formula:

$$Q = -\frac{P}{2} e^{-\lambda x} (\cos \lambda x)$$

The shear force will be maximum below the point load with negative sign and the next maximum will be from a distance of $3\pi/4\lambda$ from the point load, but with a positive sign.

$$Q_{\text{maximum}} \big|_{\lambda x = 0} = -\frac{P}{2} = -100kN$$

$$Q_{\text{maximum}} \big|_{\lambda x = \frac{3\pi}{4}} = -\frac{200}{2} \times e^{-\frac{3\pi}{4}} \times \cos\left(\frac{3\pi}{4} \times \frac{180}{\pi}\right) = 6.71kN$$

All these positive and negative signs discussed so far are for $x > 0$ or when the point of interest is to the right of the load. If the same are to be determined at a point which is to the the left side of the load, then the sign convention will change for different quantities. The sign for deflection remains unchanged irrespective of whether the point of interest is to the left or right of the load. In other words, the deflection at two points, equidistant from the point load (one to the left and one to the right of load) would be equal with respect to both magnitude and sign. This phenomenon is true for bending moment also which is evident from the first slide shown.

In case of slope and shear force, the sign will change with the point of interest being on the right side or left side of the load. From the figure shown in the first slide, it is clear that the shear force is negative if the point of interest is to the right side of loading and is positive for the opposite.

Consider a point which is to the right side of the point load and is at a distance of x units from the load. The deflection at this point would be represented by w_x and if a point is to the left of the load and x units away from it, it would be appropriate to represent the deflection at this point by w_{-x}. The other interested quantities can also be defined by the same logic: M_x and M_{-x}; θ_x and θ_{-x}; Q_x and Q_{-x}. So, in a nutshell, the essence of the discussion so far is:

$$w_x = w_{-x}$$

$$\theta_x = -\theta_{-x}$$

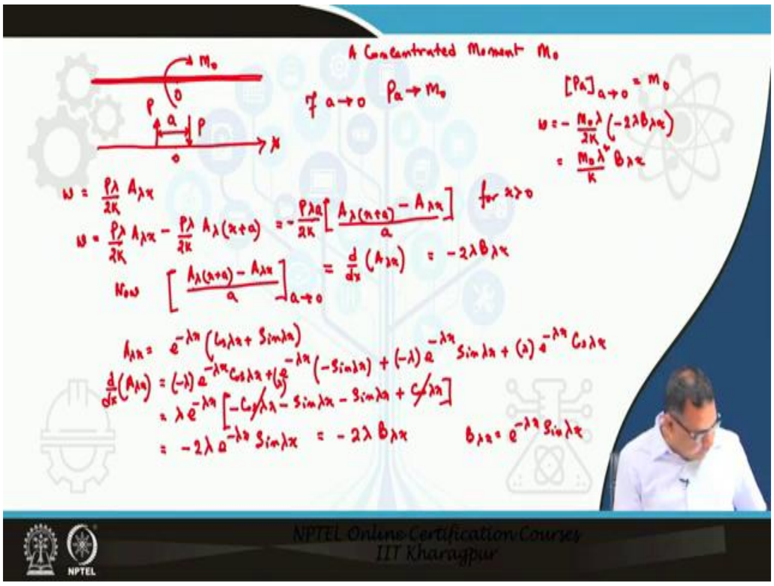
$$M_x = M_{-x}$$

$$Q_x = -Q_{-x}$$

So, the deflection and bending moment would have the same magnitude and sign irrespective of being on the left or right side. Whereas, the slope and shear force would although have the same magnitude, would differ in sign.

It should be noted that these signs are true only if a point load acts on the beam and this may change for other loading conditions.

(Refer Slide Time: 14:50)



Now, let us consider a case where an infinite load resting on soil is subjected to a concentrated moment M_0. In the previous case it was subjected to a concentrated load P, but now the load is a concentrated moment, M_0. Consider that this M_0 is acting on the infinite beam at a point O. A concentrated moment can be considered as a combination of two forces acting at a distance (say, a units) in opposite directions. For this case, the moment about point O would be P × a. If this 'a' tends to 0, then (P × a) tends to M_0 because this moment will also become a concentrated moment then.

To find the deflection for this loading condition, consider a point on the infinite beam which is at a distance of x units from one of the point loads. So, the distance between this point and another load would be (x+a). Now considering the first load, the deflection would be equal to:

$$w = \frac{P\lambda}{2k} A_{\lambda x}$$

Now, consider the other load, which is acting in the opposite direction to that of the first. As already discussed, the sign of deflection is irrespective of whether the point of interest is to the left or right of the load. So, the deflection effect due to the other load would be negative and the combined effect of both the loads would be:

$$\Rightarrow w = \frac{P\lambda}{2k} A_{\lambda x} - \frac{P\lambda}{2k} A_{\lambda(x+a)}$$

If the direction of both the loads is same, there would be a plus sign in the above expression. Multiplying and dividing the above equation with 'a' and taking the negative sign out of the bracket, we get:

$$\Rightarrow w = -\frac{P\lambda a}{2k}\left(\frac{A_{\lambda(x+a)} - A_{\lambda x}}{a}\right) \text{for x > 0}$$

As discussed already, if 'a' tends to 0, the moment tends to be concentrated. So:

$$\left(\frac{A_{\lambda(x+a)} - A_{\lambda x}}{a}\right)_{a \to 0} = \frac{d}{dx}\left(A_{\lambda x}\right)$$

$$\text{But, } A_{\lambda x} = e^{-\lambda x}\left(\cos\lambda x + \sin\lambda x\right)$$

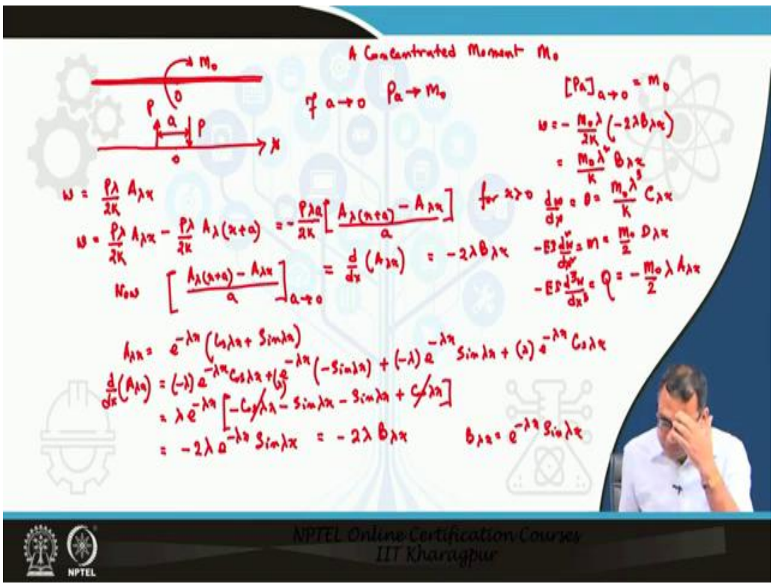
$$\text{So, } \frac{d}{dx}\left(A_{\lambda x}\right) = (-\lambda)e^{-\lambda x}\cos\lambda x + e^{-\lambda x}(-\sin\lambda x) + (-\lambda)e^{-\lambda x}\sin\lambda x + (\lambda)e^{-\lambda x}\cos\lambda x$$

$$\Rightarrow \frac{d}{dx}\left(A_{\lambda x}\right) = (-\lambda)e^{-\lambda x}\left[-\cos\lambda x - \sin\lambda x - \sin\lambda x + \cos\lambda x\right]$$

$$\Rightarrow \frac{d}{dx}\left(A_{\lambda x}\right) = -2\lambda e^{-\lambda x}\sin\lambda x = -2\lambda B_{\lambda x}$$

$$\text{Since, } \left(\lambda e^{-\lambda x}\sin\lambda x = B_{\lambda x}\right)$$

(Refer Slide Time: 25:41)



As Pa tends to M_0 when 'a' tends to 0, we can write the expression for deflection as:

$$w = -\frac{M_0\lambda}{2k}\left(-2\lambda B_{\lambda x}\right)$$

$$\Rightarrow w = \frac{M_0\lambda^2}{k}B_{\lambda x}$$

Slope at a point is a change in deflection with respect to the distance. So, by differentiating the above expression for deflection, we get slope:

$$\Rightarrow \frac{dw}{dx} = \theta = \frac{M_0 \lambda^3}{k} C_{\lambda x}$$

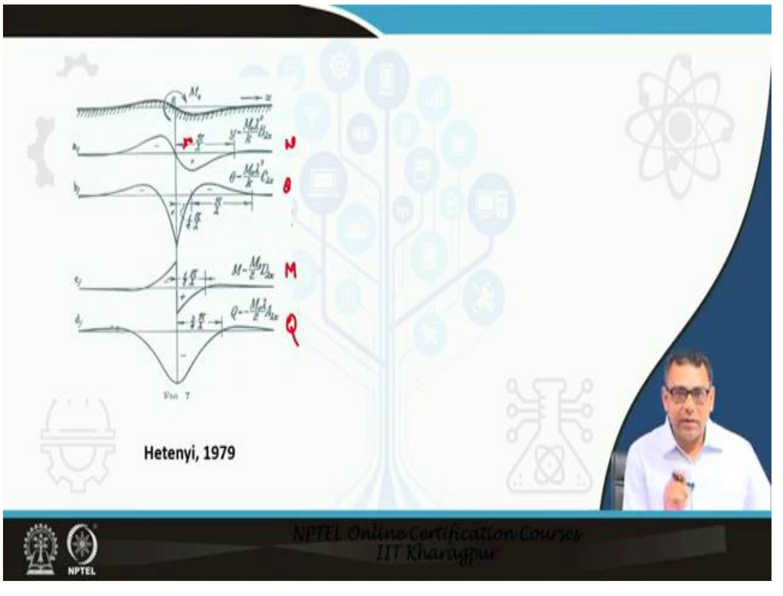
Similarly:

$$\Rightarrow -EI\frac{d^2w}{dx^2} = M = \frac{M_0}{2} D_{\lambda x}$$

$$\Rightarrow -EI\frac{d^3w}{dx^3} = Q = -\frac{M_0 \lambda}{2} A_{\lambda x}$$

These are the equations that for an infinite beam subjected to a concentrated moment. The equations for a concentrated load and concentrated moment acting on an infinite beam are very important and will be used to derive expressions for other complex loading conditions like UDL, triangular loading, etc., for infinite, semi-infinite and finite beams. So, it is better to remember these 8 equations (*4 for concentrated load + 4 for concentrated moment*).

(Refer Slide Time: 29:24)



The profile of each quantity showing the variation in each of them along with the distance from the load is shown in the slide above for a concentrated moment. The deflection would be 0 when x has the values o, π/λ, 2π/λ etc., and slope would be 0 when x equals π/4λ, 5π/4λ, etc. Similarly, the bending moment would be 0 at x = π/2λ and the shear force would be null at x = 3π/4λ.

The maximum shear force will be at the point of loading similar to that of the point load along with the bending moment and slope. All three quantities except deflection would be maximum at the point of loading when the load is a concentrated moment. The deformation would be maximum at another point and to find this, the derivative of the deflection equation need to be considered and the distance should be determined.

The prime idea of explaining the cases of concentrated load and concentrated moment is because they are very important to solve for beams under other loading conditions and to stress the need to remember these 8 equations. Thank you.