Structural Analysis 1
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Lecture 12
Analysis of Statically Determinate Structures: Method of Virtual Work

Hello everyone, welcome to the third week of this course. Last week we have seen how to analyse statically determinate trusses using method of joints and method of sections. And this week we will be learning how to determine deflections in truss. Now though as far as application is concerned we will only apply the concept that we will learn in this week to truss but the concept is the widespread and has a wide application in structural mechanics.

What we are going to do is we are going to learn analysis of statically determinate structures, method of virtual work and more precisely it is the deflection determination of statically determinate structures using the method of virtual work.

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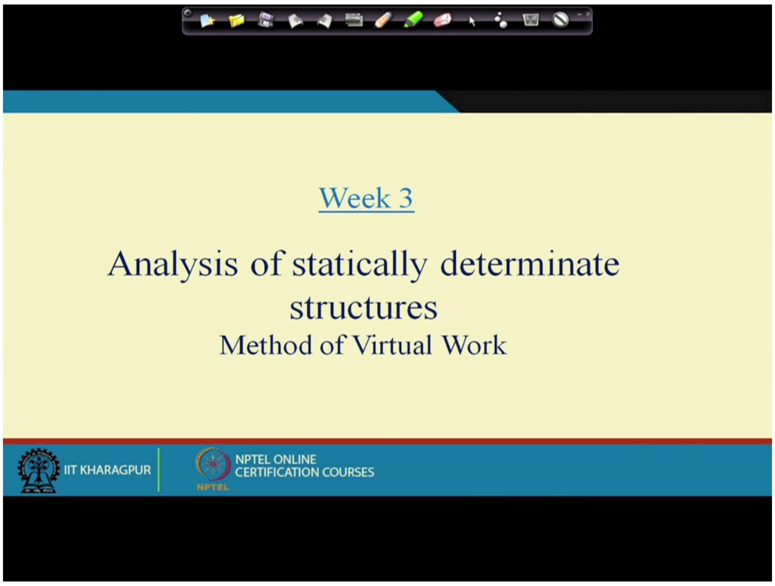

Again as I said we will introduce the concept of virtual work and apply it to determine the deflection of trusses but the method has a very wide application and it has different interpretations and there are many methods which are formed with this concept. So we will just learn the concept and apply to trusses and for beams and frames we will be doing in subsequent lectures.

And the other forms of interpretation of this method that we will also learn in subsequent or discussing it in the subsequent weeks, okay. So as today is the first lecture of this week we will try to understand what is virtual work and what is the principle of virtual work, the concept behind the principle of virtual work, okay. Now before we actually discuss what is virtual work let us see what is work that we all know.

What is work done by a force? Again we know that if we apply a force at any point A and the point moves to A dash and the vector is dr. These all are written in a vector form. Okay, dr is the vector. Then the work done by the force is, W is equal to integration of F dot dr and that integration has to be done over the entire path.

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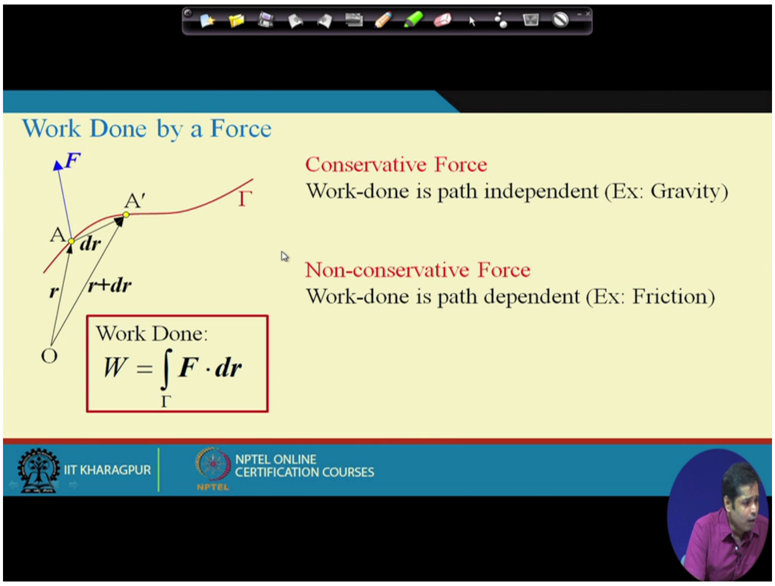

Now depending on whether this work done depends on path or not, we have two kinds of forces, one is conservative force. We say that the work done is path independent. The example of conservative force is gravity force. And then the non conservative force where the work done depends on the path. So it is not path independent. The example of non conservative force is friction.

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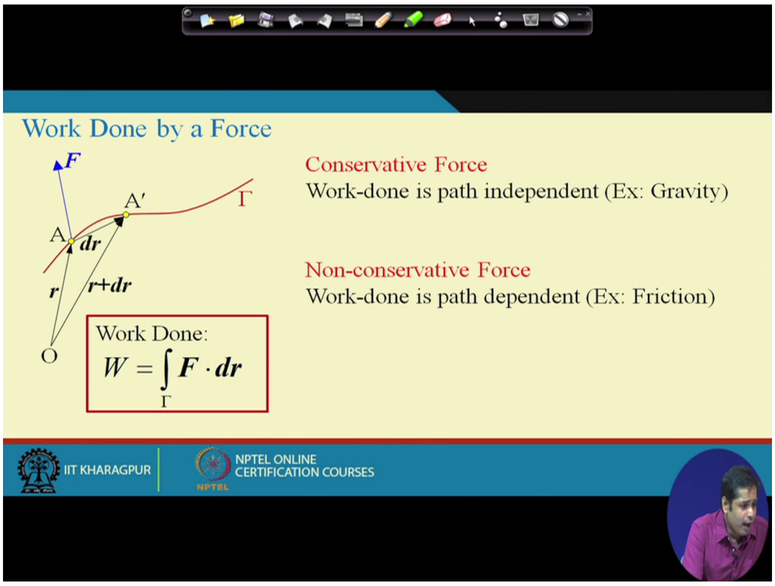

Now but in this case we will be talking only about elastic force. The elastic force is conservative and you see this is the load deflection curve, so if you take a spring and apply a load P then the deflection is x and this deflection and then this load they are related. If it is linear elastic they are related like this, the linear relation in a load deflection, okay.

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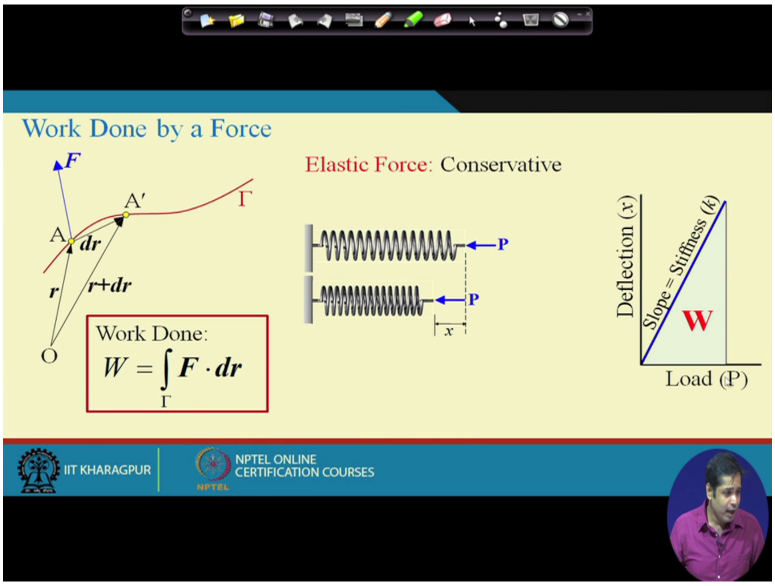

Now so the elastic force is conservative force. Now if you remember we started our discussion with equilibrium. What equilibrium says that if we take a body and there are some external forces acting on the body, as a reaction to that there will be some internal forces in the body and when the object is at equilibrium state then the total external load should balance the internal forces. The external force is equal to the internal forces. That is equilibrium, right?

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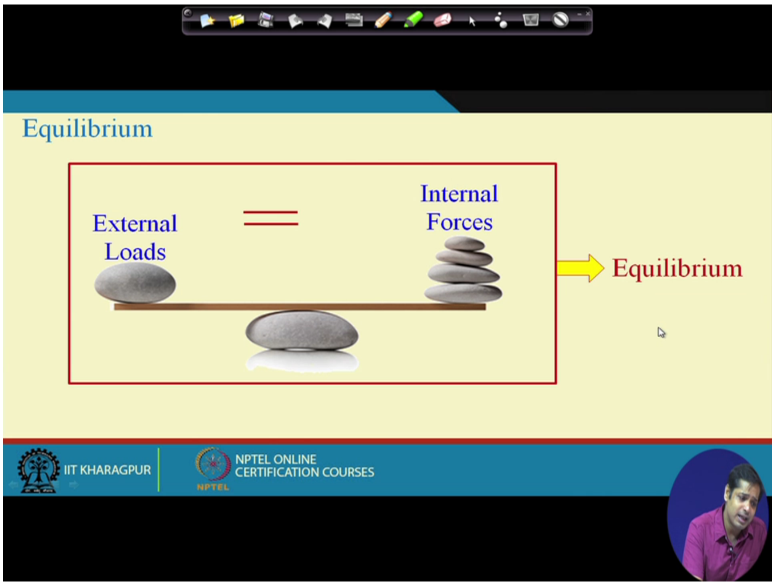

Now suppose we are (inter) interested in force but suppose these equilibrium or this entire concept if we try to understand, instead of external and internal forces if we try to understand through external and internal work. So if we take an object, apply some force then actually that object will undergo some deformation.

Means the force is doing some work and then as a reaction to that there will be some internal forces and this change develops in the body and there will be some internal work as well. So now the (con) question is how this external work and the internal work they are related to each other? And based on their relation what you understand or what we interpret from their relation? So that is the core objective of todays class, okay.

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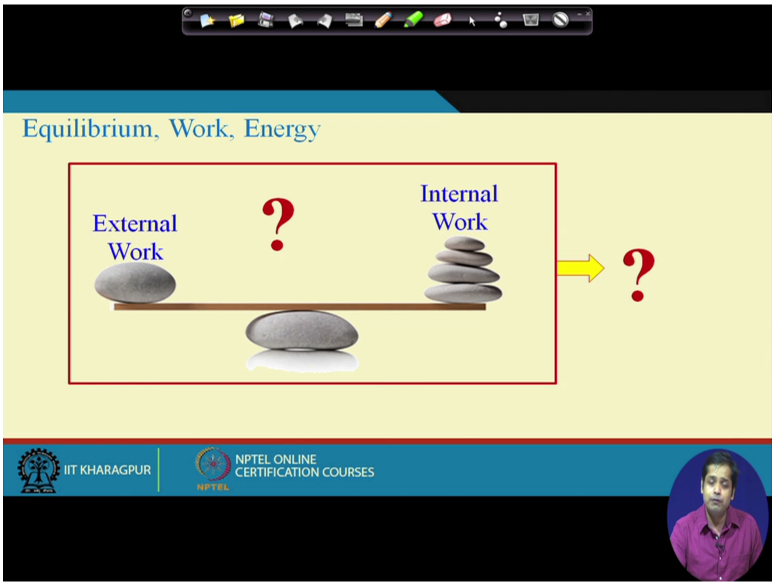

We will see how this internal and external force, not today, in this weeks class we will see how the internal and external forces are related to each other and once we have that relation what information they give about the behaviour of the structure, right? Now we know what is equilibrium, right? That is known to us. Now what we say, at the equilibrium the internal and external force, they balance each other.

Now you want to see what happens if we try to understand the same thing instead of forces in terms of work. Let us take the same example.

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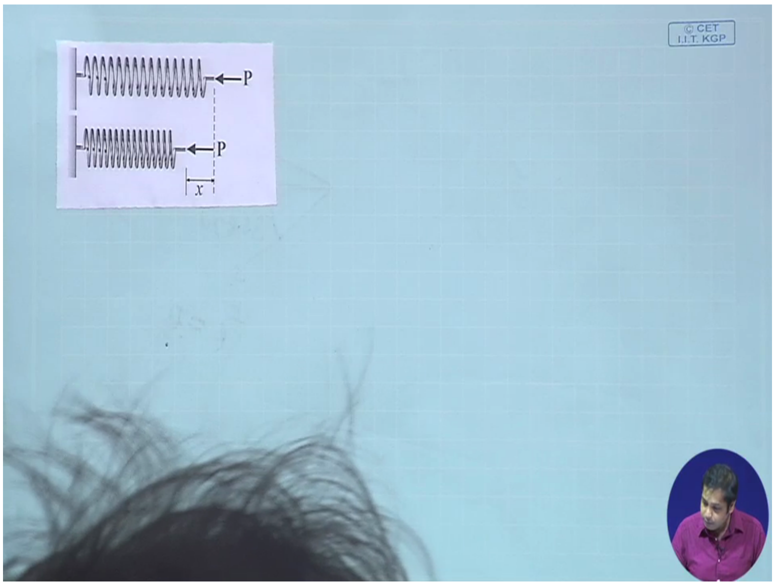

This is a spring subjected to a force P and displacement x, okay. Now then what is the work done by this force? So external work will be P into x. That is the work done. Now suppose this spring constant is K which is the stiffness of this object. Suppose this is K. Stiffness of this object is K.

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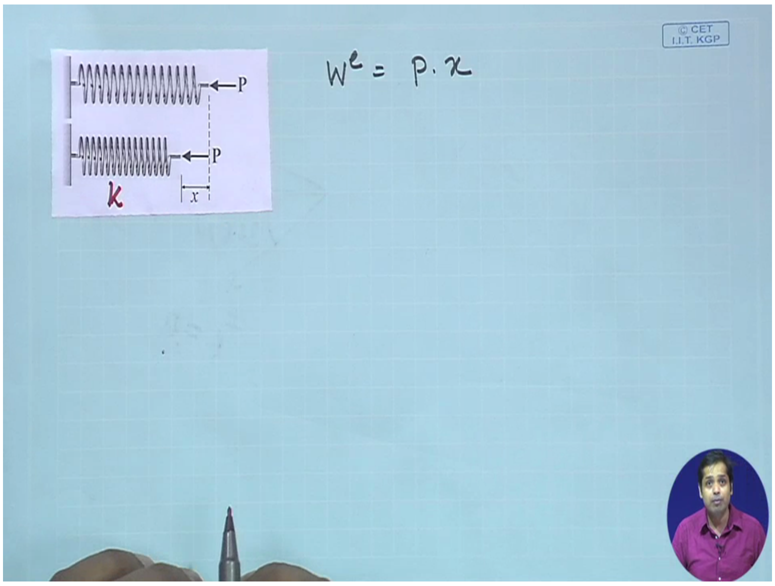

Now if it is x then the force generated in the object is Kx. So if we take a section like this and draw the free body diagram of this section, this free body diagram will be, if this force is P and then there is a force generated in this object which is K into x, okay. And K is stiffness or stiffness of the spring or spring constant, okay.

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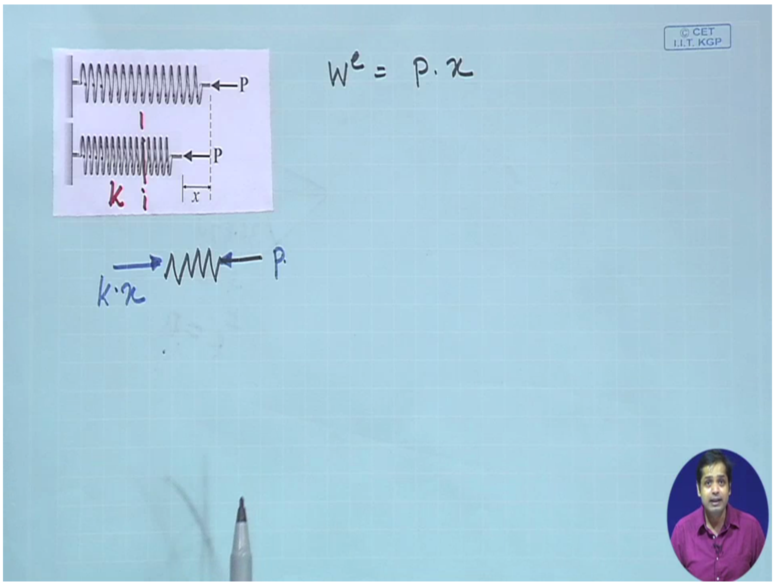

Now we do not know whether Kx and P, if it is in equilibrium then we say that P is equal to Kx. Summation of forces in horizontal direction is zero so which gives us P is equal to Kx if that is in equilibrium, right? P is equal to Kx if this is in equilibrium, right?

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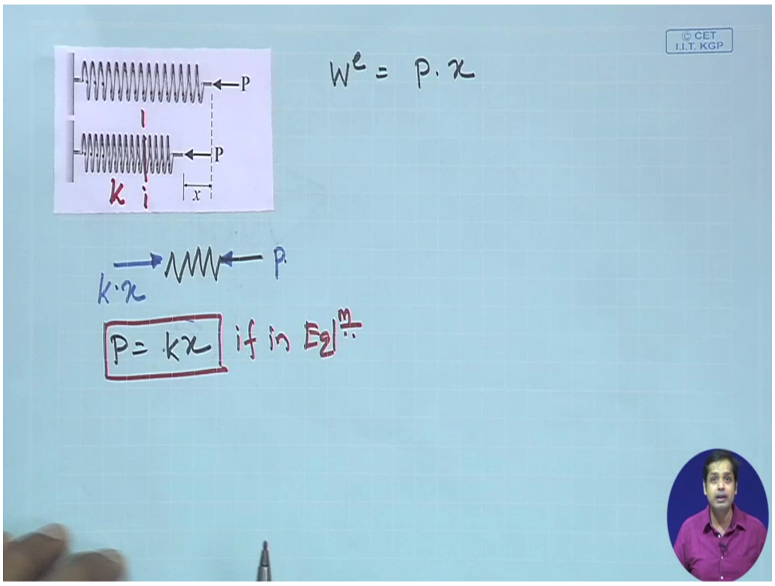

Now so this is external work. Now just now we have seen that if we draw the load displacement curve, this is the load and this is the displacement curve. And this displacement curve is linear and the area under this load displacement curve will give us the internal energy.

And what is the area? The internal energy will be, in this case if we see that half into x is the displacement and the force F is Kx because that is the internal force generated in this spring. This is Kx, we all know this. This is half into Kx square, right?

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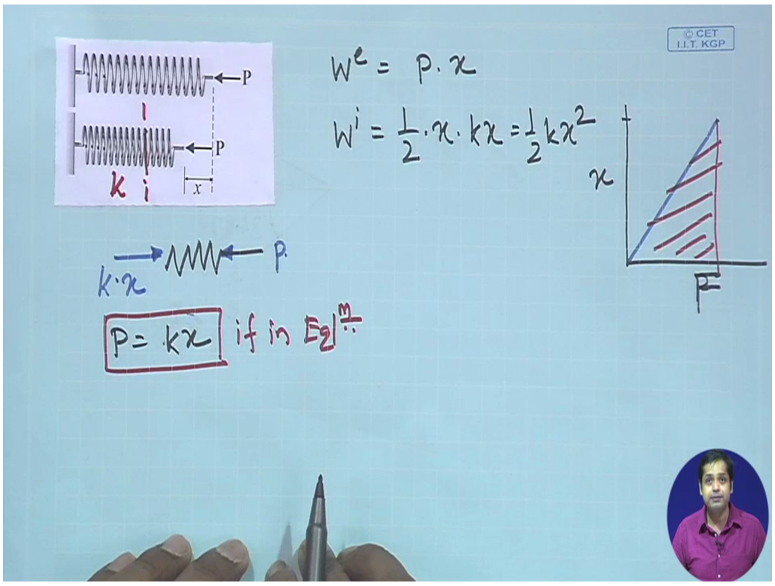

Now then if we define a potential the total potential energy becomes pie is equal to Px p into x minus half into Kx square, right? Now this is the total potential energy of this of the system, okay. Now we define the potential energy like this.

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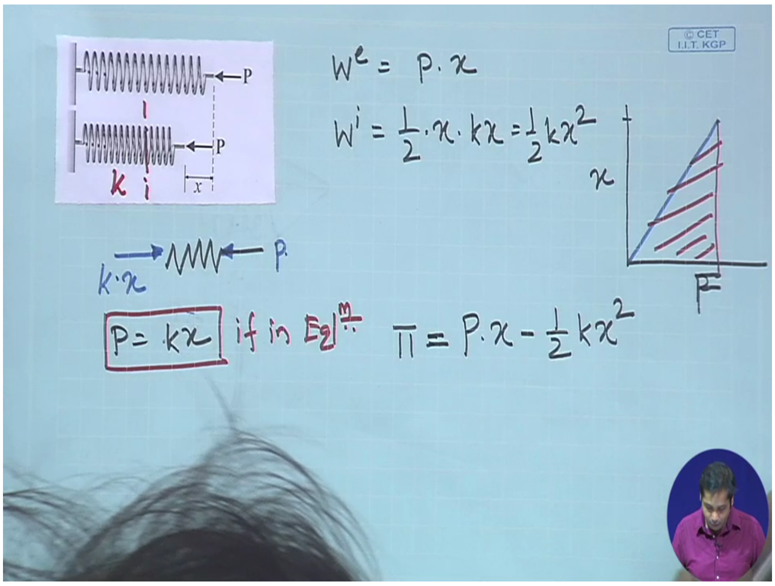

Now what we do is let us give a small perturbation in the (displace) displacement field. Assume that the force acting on this P is constant and the stiffness of this spring, K is also constant. The only variable we have is x. Give a small perturbation in x. Small perturbation, by small perturbation I mean now this is the equilibrium position, right?

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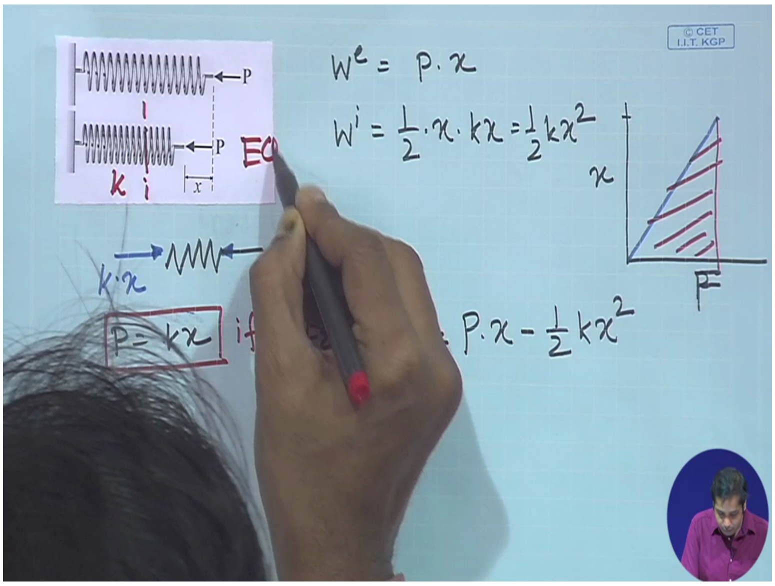

Now at the equilibrium position this distance is small x. Now at this equilibrium position suppose now we give a spring and this is force P and suppose this distance at equilibrium position was x. Now suppose this distance is x plus delta x, okay. So we give a small perturbation of delta x in the displacement field, okay.

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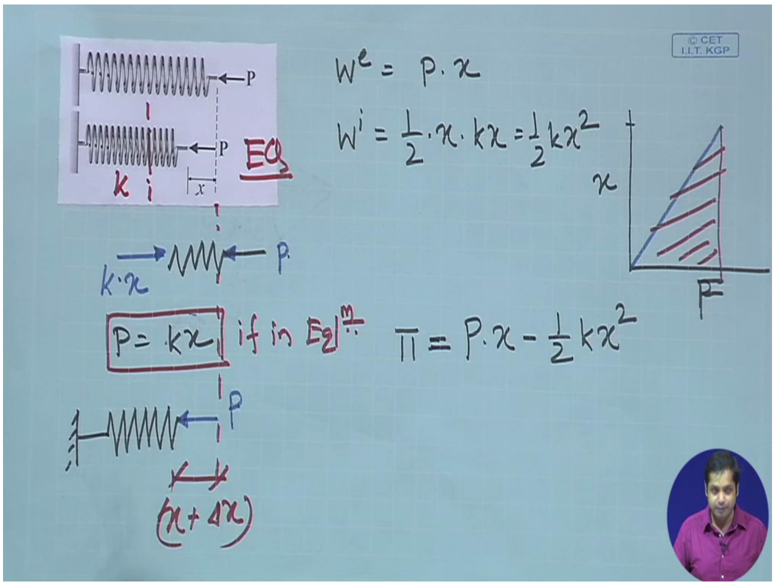

Now what would be the corresponding potential? For total potential energy, pie will be P into x plus delta x and then minus half of K into x plus delta x square, okay. This will be the total potential.

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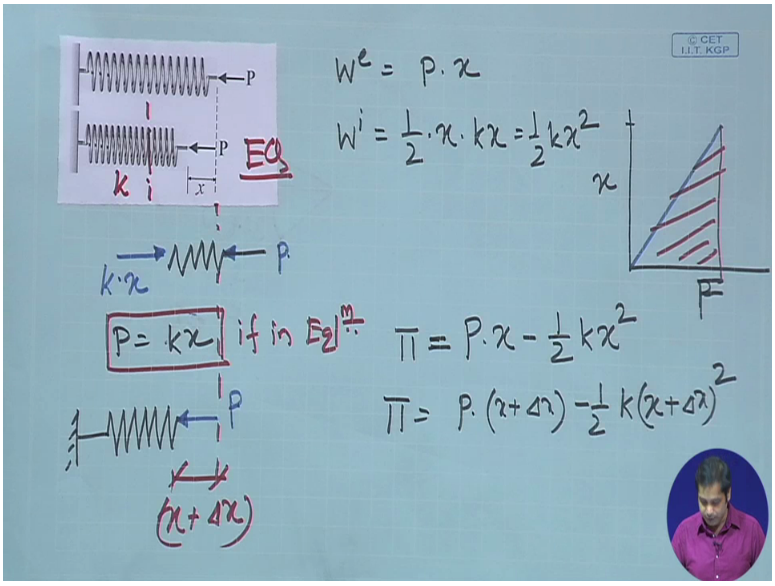

So this is equilibrium state. So it is zero and this is 1, okay.

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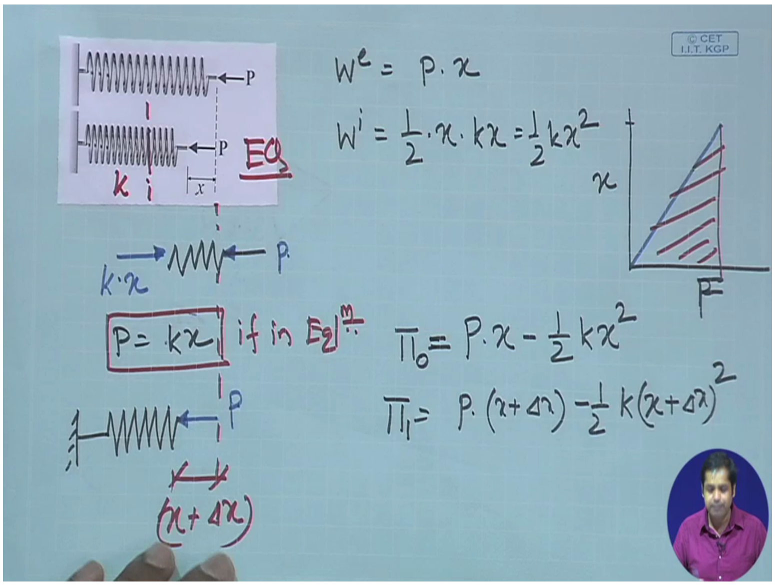

Now then what is the change in this potential energy? What is the variation in the potential energy? The variation in the potential energy is pie 1 or delta pie if you can write delta pie that is equal to pie 1 minus pie 0, right? Pie 1 minus pie 0. This is equal to pie 1 minus pie 0 and that we have is P into x plus delta x minus half of K x plus delta x square and then minus P into x plus half of K x square.

And if we take this Px and this Px will cancel each other and then we have finally K into or rather like this, P minus Kx into delta x plus half K delta x square. So this is delta pie, okay.

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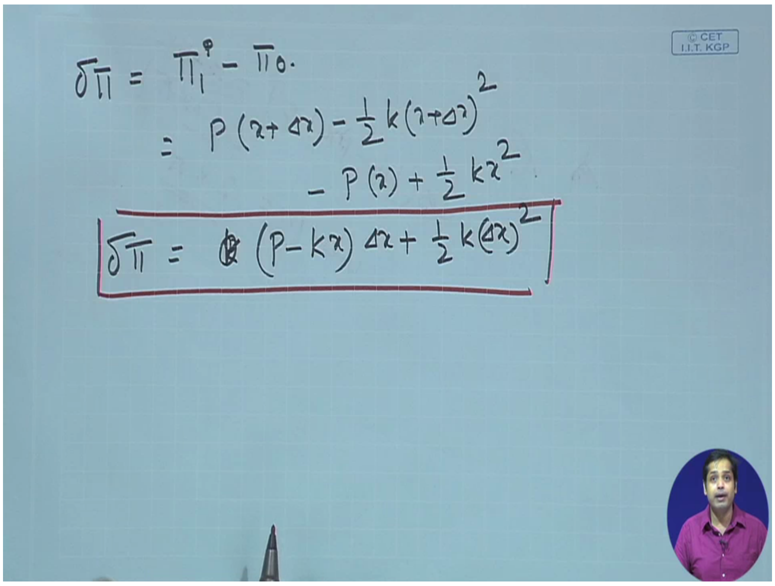

Now this variation in displacement field delta x is very small. This is one of the very important assumption. Delta x is very small. If delta x is very small then we can neglect this quadratic term and then we can say del pie that is equal to P minus Kx delta x, okay. Now so this is thing we have.

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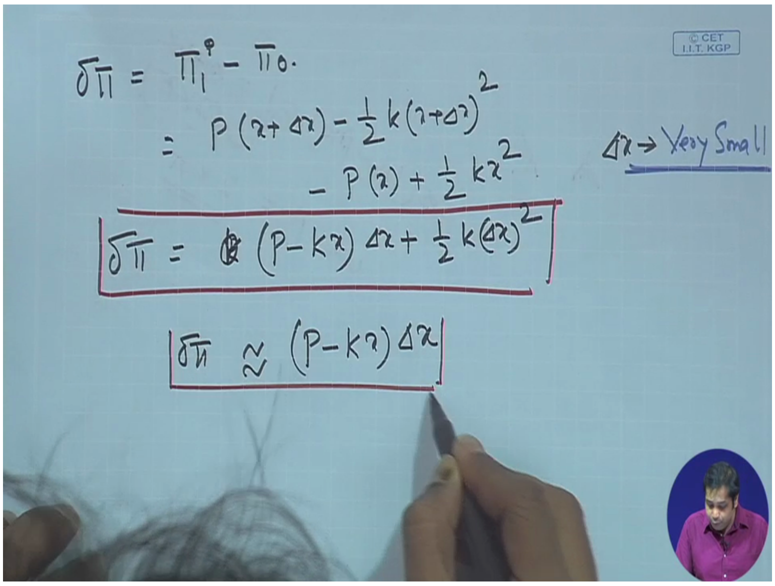

If you remember the equilibrium condition if the object is in equilibrium then P minus Kx is equal to zero. So this P minus Kx is equal to zero. If P minus Kx is equal to zero then del pie is equal to zero. Means what it says? It says that if the object is in equilibrium then variation in total potential energy that variation is due to the small perturbation in the displacement field. Now the variation in the total (po) potential energy, the first variation in the total potential energy is zero, okay.

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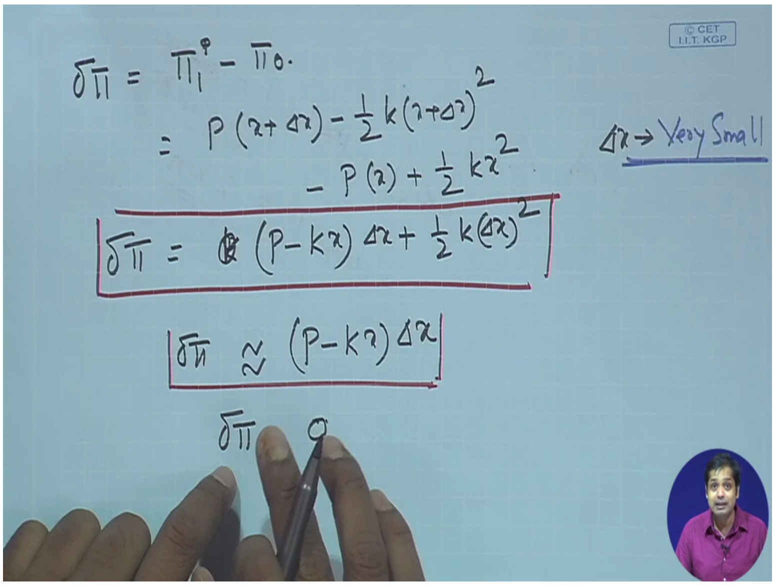

Now if you see the reverse, reverse means if we assume that delta pie is equal to zero, okay. If delta pie is equal to zero then P minus Kx into delta x is equal to zero and delta x is any arbitrary perturbation. It is not zero which gives you P minus Kx is equal to zero which is equilibrium. So this tells that object is in equilibrium.

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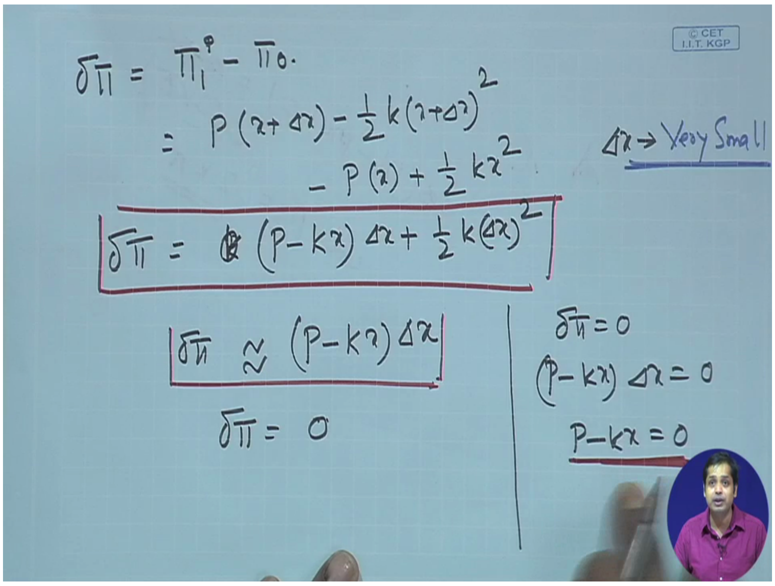

Now there are two things here. One is if it is in equilibrium then del pie is equal to zero and if del pie is equal to zero then it is in equilibrium, okay. So if the object is in equilibrium then it minimises, then the small variation in the prototype potential energy that becomes zero. And if that becomes zero then the object is in equilibrium. We can say that the object is in equilibrium. But only criteria is the perturbation the delta x has to be very small.

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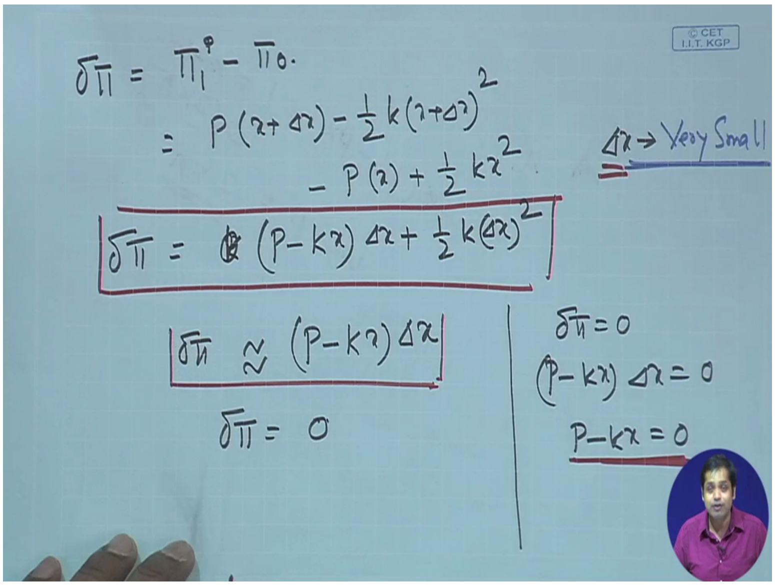

Now what I am trying to do is demonstrate of course through a very simple problem that the relation between the state of equilibrium and the energy. Now from this observation what we can conclude or what we can state? Now we have two things. One is it is equilibrium which says the external forces is equal to internal forces and this we want to actually see what happens and how this external internal work are related? Now how these two states are related to each other? This (conce) definition of state and this definition of state, how they are related to each other?

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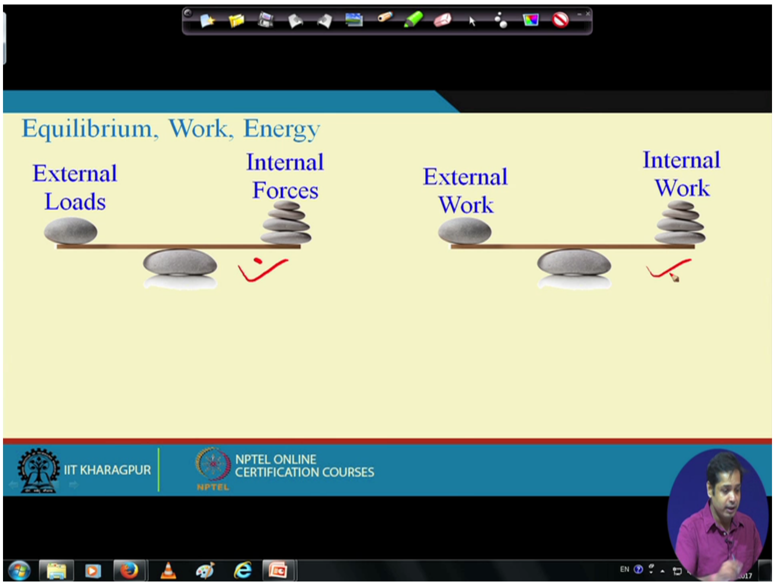

Now what we have from this example? The first is at equilibrium state structure minimises its total potential energy. Now this also you can check here it is the first variation.

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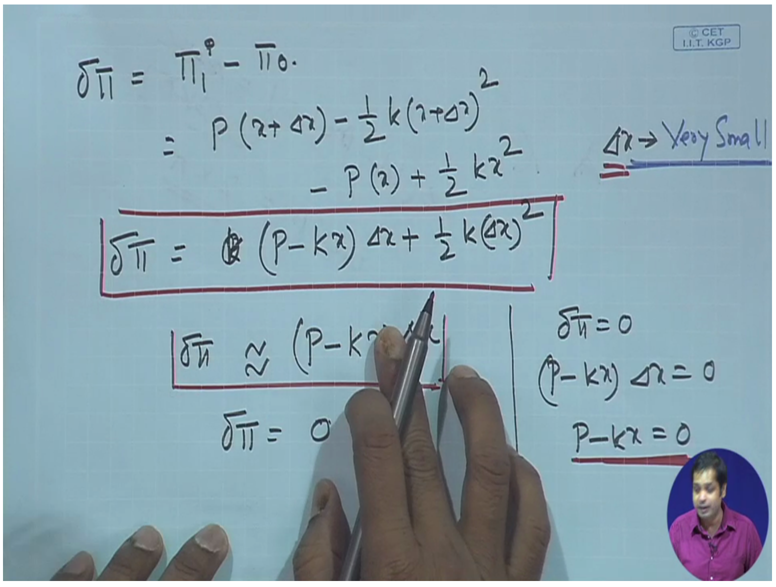

Now if you take the second variation then the second variation you will get the similar. First if the potential energy has extreme value either maximum or minimum its first variation is zero. And second variation is also very similar to the second derivative if the second variation is greater than zero then we say that it is it is minimum. And this is first variation, sometimes written as like this.

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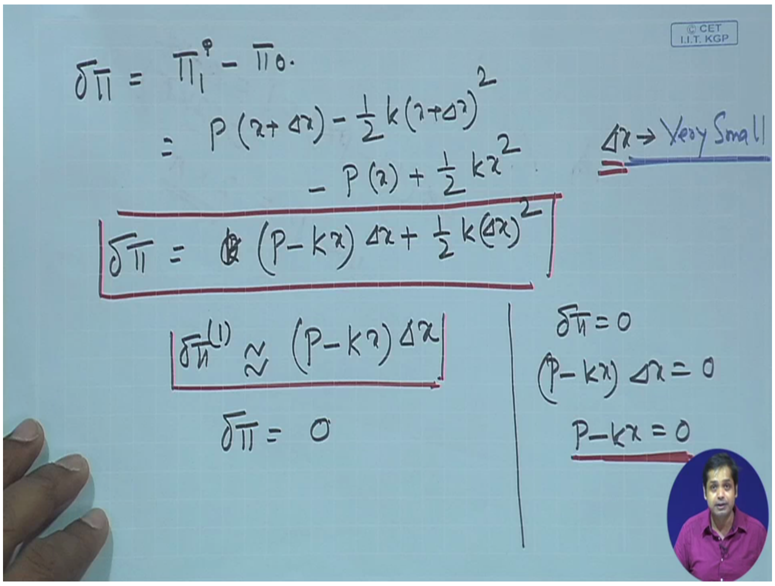

Now similar to this if we write the second variation the second variation will be half of K into delta x. You can check this, okay. The way we defined second derivative, similar way you can get the variation.

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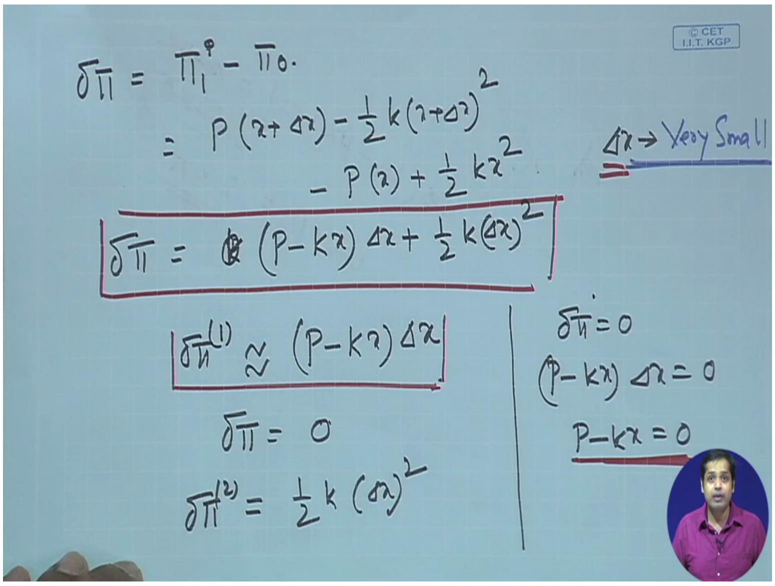

Now delta x is always positive. So this is always greater than zero, right? It means that it minimises the potential energy so which is the first statement that at equilibrium state structures minimises its total potential energy. And we have already learnt it in physics, right?

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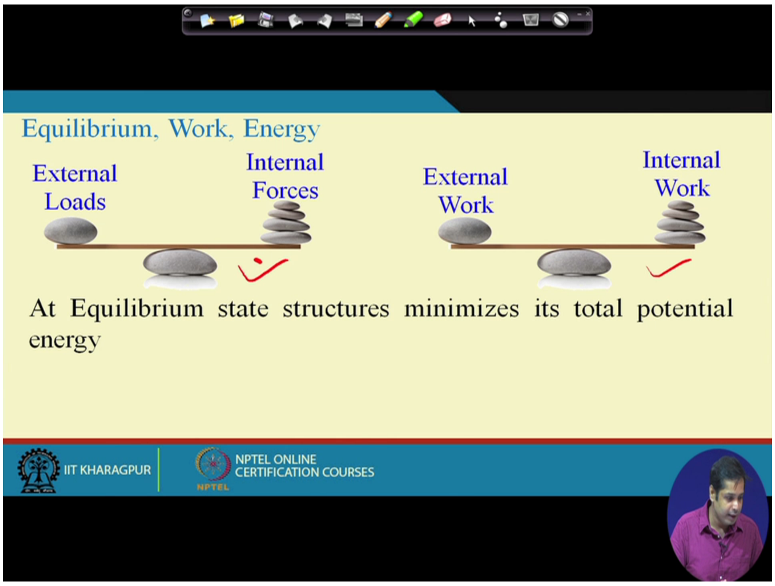

Now second thing is when a structure is in equilibrium state a small variation of potential energy vanishes. Now in this case the variation is due to the change in or the perturbation in the displacement field, okay. Now what actually is the concept of virtual work or principle of virtual work? Principle of virtual work is based on these two observations that you have just now seen. That they constitute the basis for the principle of virtual work, okay.

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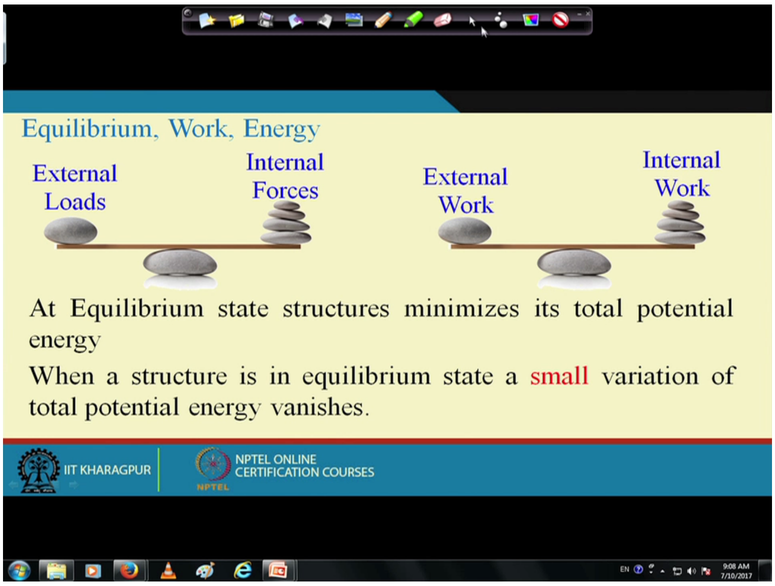

Now before we actually state the principle or derive the principle of virtual work, understand the principle of virtual work, its principle, let us first understand what is virtual work? We know what is work? Now let us understand what is virtual work, okay. Now you see suppose take any object which is subjected to some force, okay. F1, F2, F3.

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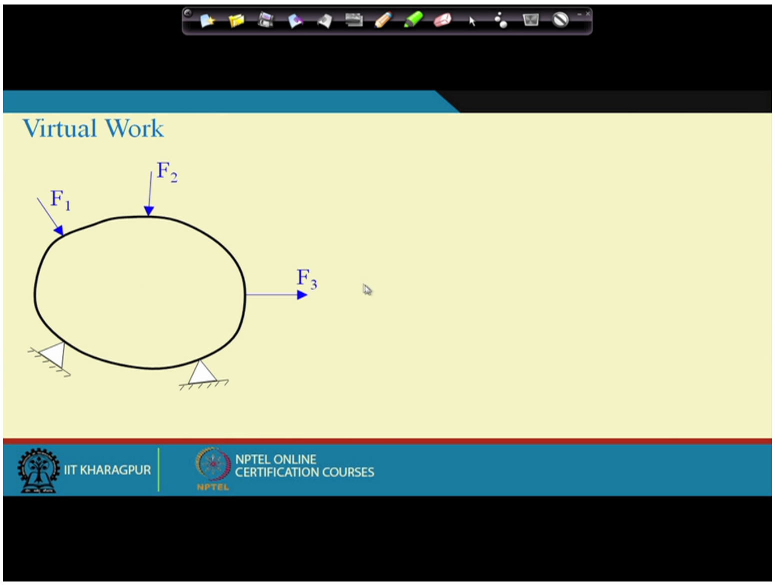

Now what happens suppose if I give some displacement, okay. Now this displacement in this object is not due to this applied force. This is independent of the applied forces, okay.

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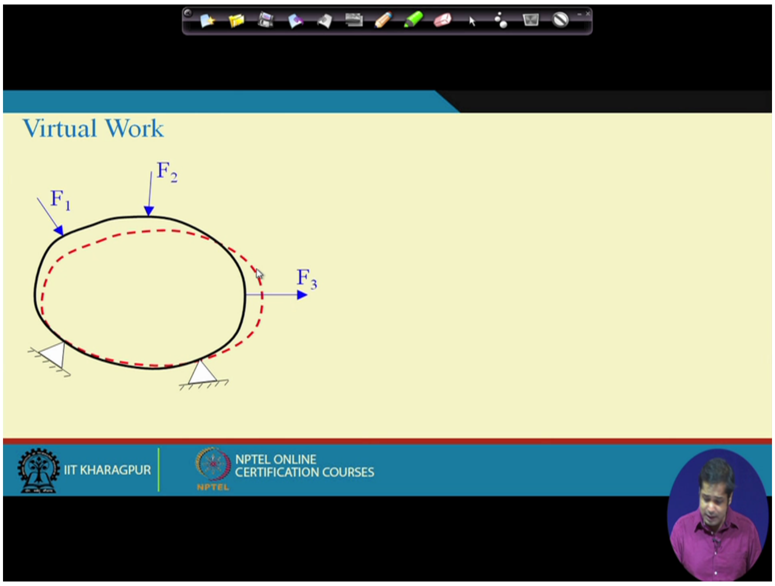

For instance again it is the similar (examp) thing here. This displacement that we gave delta x, this is not the actual force. This is independent of the actual forces, okay. But they have some relation between the actual forces. What relation? I will discuss that. But similarly to delta x here we give some perturbation some displacement, okay. And this displacement is not the actual or this deformation is not the actual deformation of the structure.

This is virtual displacement. The effect is an imaginary displacement we give to the structure, okay. That is why it is shown in dotted line. Now this displacement is called virtual displacement, okay.

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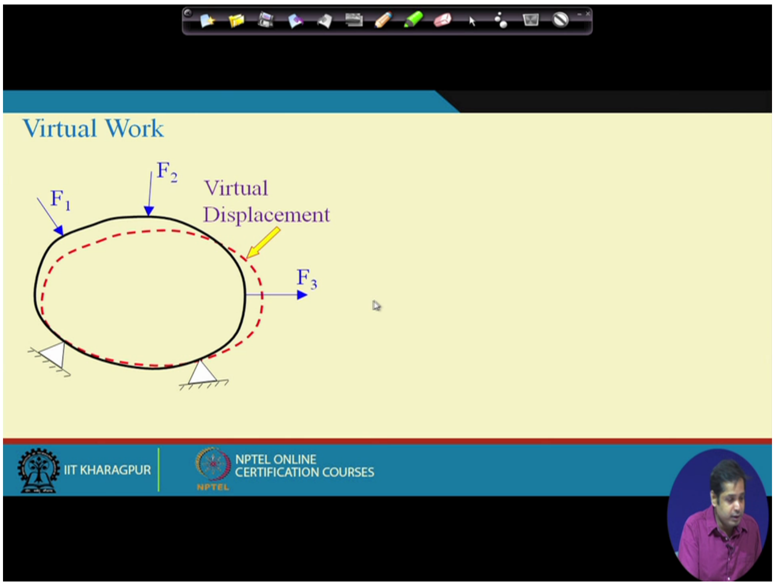

Now suppose at the action of this forces the corresponding virtual displacements are D1, D2, D3, okay. Then put a star here, the virtual displacement. When I say it is independent of the force it does not mean that any deformation or any kind of displacement can be virtual displacement. There are certain criteria to be satisfied which we will discuss shortly, okay.

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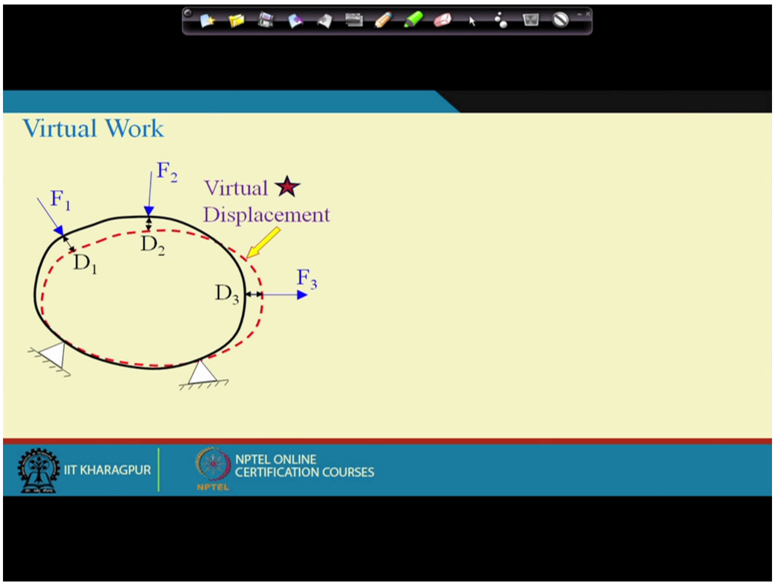

Now had it been then what is the work done? See had it not been virtual displacement. Suppose had it been the actual deformation in the object because of this external forces then what would have been the work done? Work done is F1 into D1 plus F2 into D2 plus F3 into D3. That would have been the work done. But now this work done is due to the applied forces. But the object is moving through the deformation in the object is not the actual deformation.

Now then this work is called virtual work. So what is virtual work? Virtual work is the work done by the actual forces acting on the body moving through virtual displacement. So this is the work, this is the actual body and this is the actual forces acting on the body.

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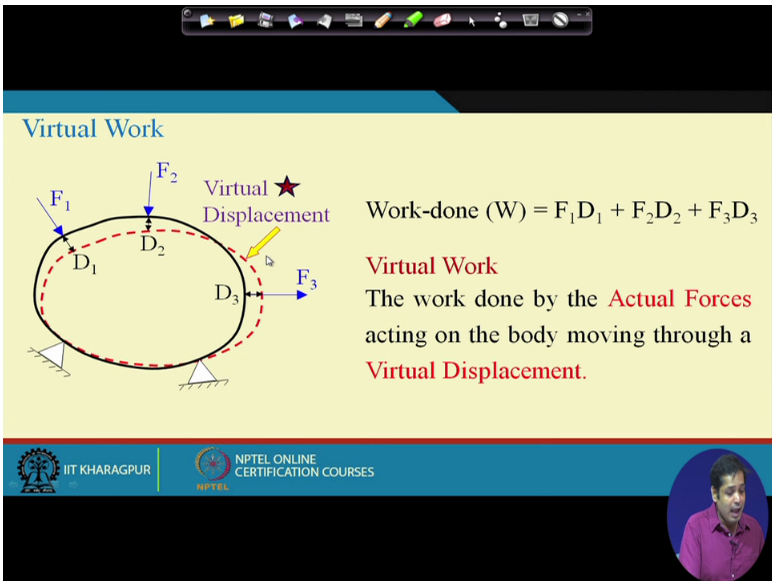

And then if we give some virtual displacement to the body then the product of the force and associated virtual displacement is actually measure of some work, okay. And this work is called virtual work, okay.

I am saying it is a product of the force and associated displacement because it is in the product but I am not saying it is work done by the force because this deformation is not due to these forces. So this F1, F2, F3, this is called force field and D1, D2, D3, this is called displacement field.

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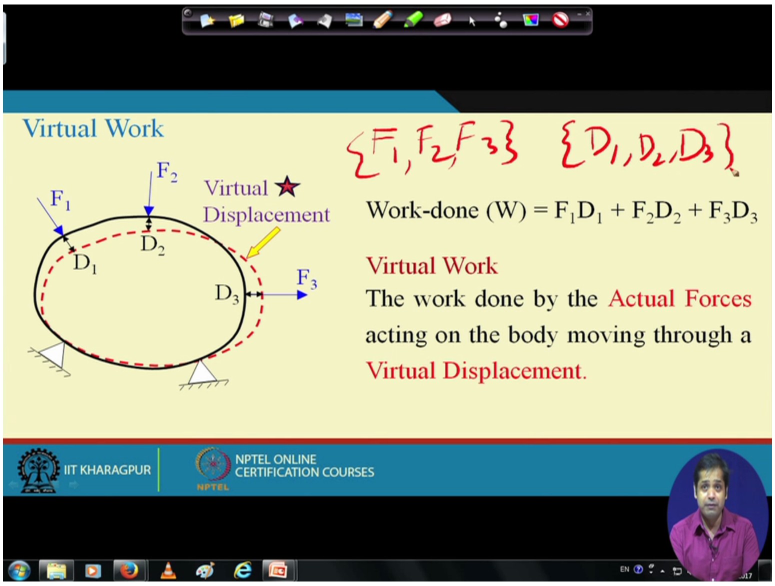

Now this force field and displacement they do not have cause of a relationship. The displacement field is independent of the force field. This displacement is not due to this force, okay. This is virtual displacement, okay. Now then this force field and D1 is called the conjugate displacement field. Since it is a virtual displacement but this displacement is conjugate to this force field, okay. Now so this is virtual work, right?

Now just to give you some more examples so that you can relate these examples to the kind of problems that we are addressing in this course. Again take a simply supported beam which is subjected to a concentrated load at any arbitrary point in this and then give a small displacement which is shown in dotted line and at this point associated displacement is D. Then the virtual work is F into D, okay.

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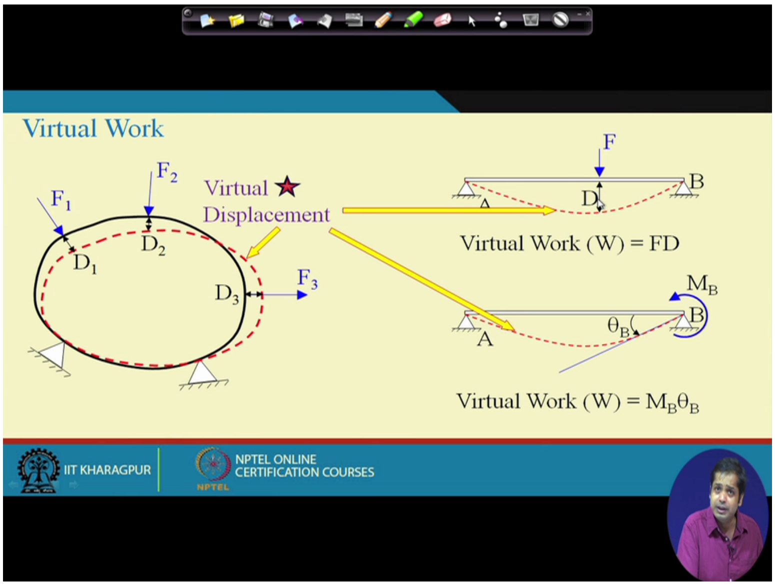

So it is the actual force working on the body. The F is the actual force acting on the body and D is the virtual displacement. Similarly you take simply supported beam but it is subjected to some moment at B that gives a virtual displacement like this and this is again virtual rotation at theta is theta B at B.

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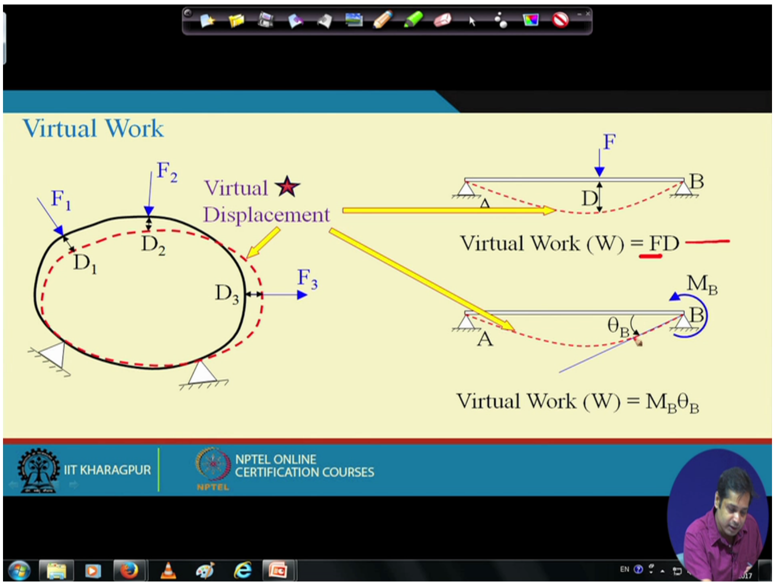

Then the virtual work done is MB into theta B, okay. Had it been the actual deformation due to this force or had it been actual deformation due to this moment, this work would have been the actual work. But since if they are not actual deformation, they are virtual deformation that is why these works are called virtual work, okay.

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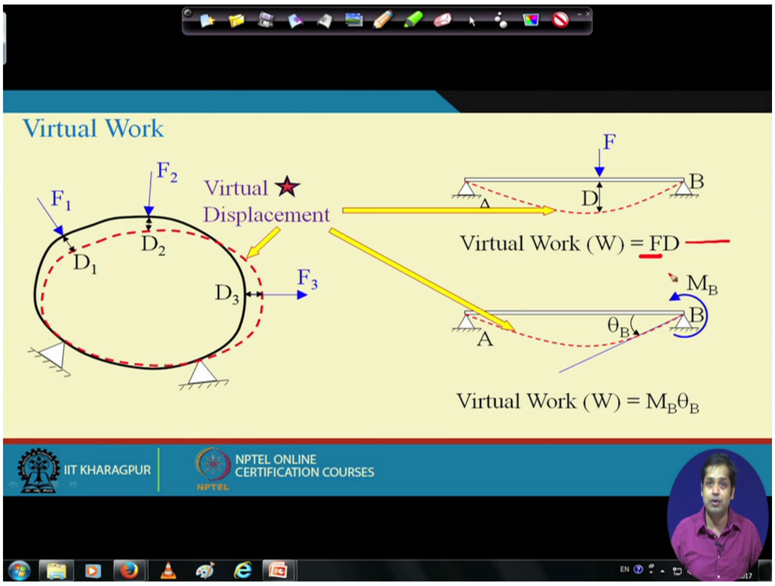

Now as I said any deformation cannot be virtual displacement. Virtual displacement should satisfy certain criteria and that criteria is compatibility criteria. For instance suppose if we take a truss, okay. This is a truss and okay, suppose these are three members. So this is simply supported and this is roller support. This is statically determinate structure, okay.

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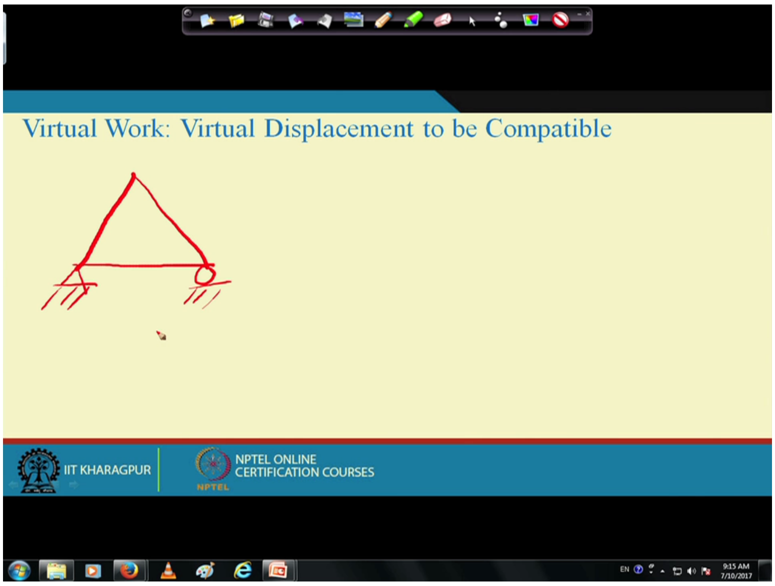

Now it is subjected to some load say it is here. Now we want to give some virtual displacement. Now if we give a displacement like this, this is compatible displacement, okay.

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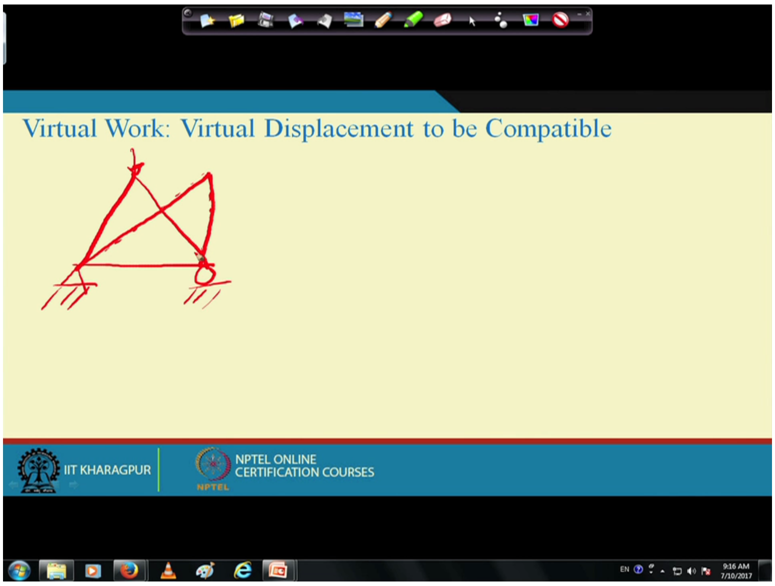

But if we give a displacement like this, the virtual displacement. Suppose this is again roller. If we give a virtual displacement like this. This is member A, B and then C, suppose this is C. This is member A, this is member B this is point B and suppose member C is this. So this is again B.

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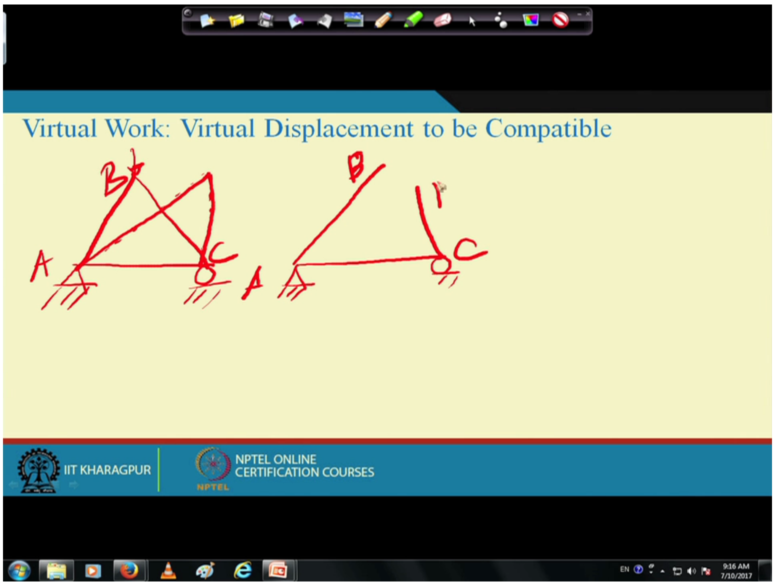

So member AB is this and member BC is this. So this is not compatible because compatibility condition at B is not satisfied, okay. Because whatever maybe the deflection, the point B is an intersection of AB and BC. There should not be any discontinuity at point B. So this cannot be a virtual displacement.

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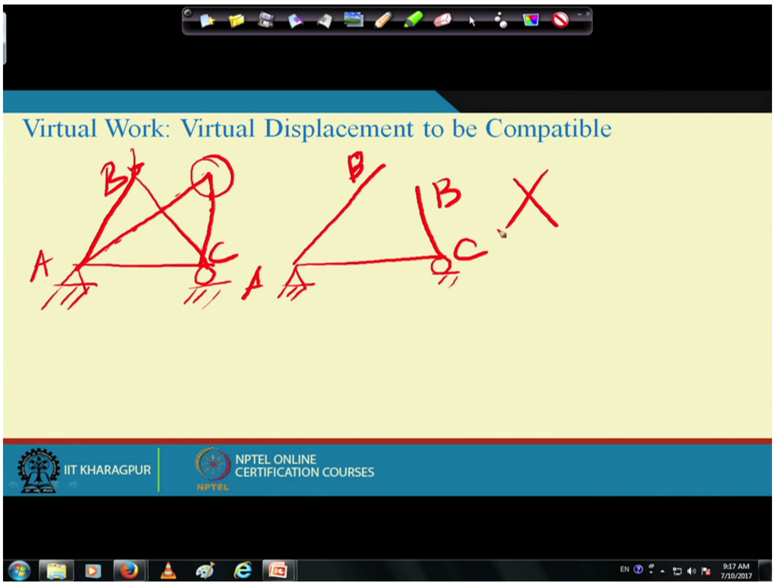

Similarly if we take a simply supported beam which is subjected to say some loading. Now this can be taken as a virtual displacement but if I give a virtual displacement like this, again this is simply supported, we cannot give this as virtual displacement, okay.

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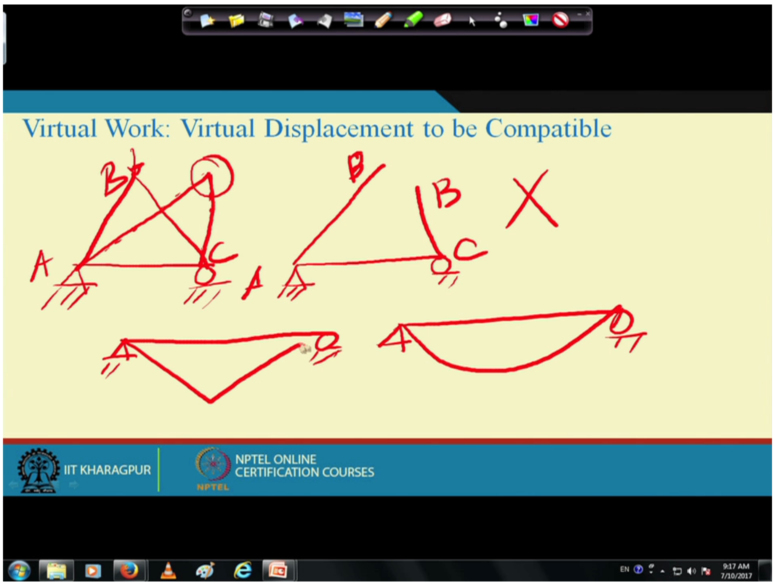

Because displacement should be not only continuous at the slope in subsequent which we derive the equation of this elastic curve and we will see that slope needs to be (co) continuous at every point, okay. And this is compatibility condition.

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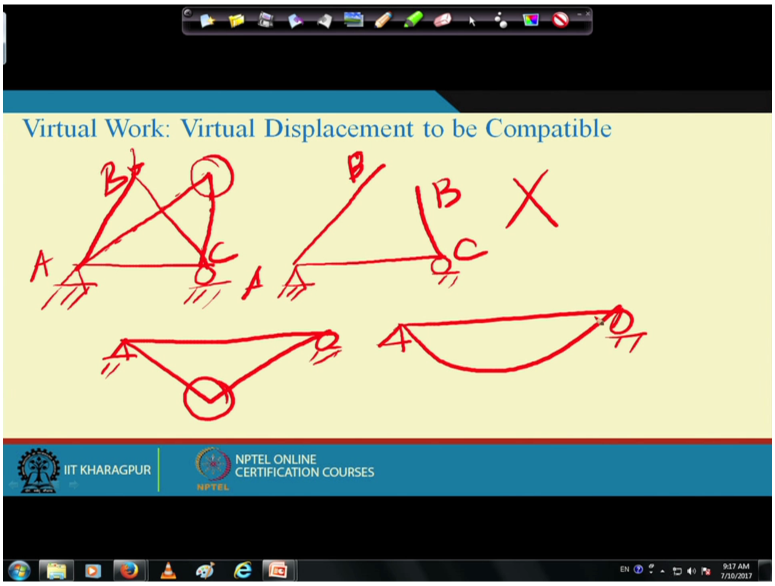

But suppose again this is a (con) simply supported beam and this is a roller support. We can give this as virtual displacement. Means this is A and this is B. So point B goes to B dash and this is delta. So we gave a virtual displacement of delta at point B. This is acceptable virtual displacement.

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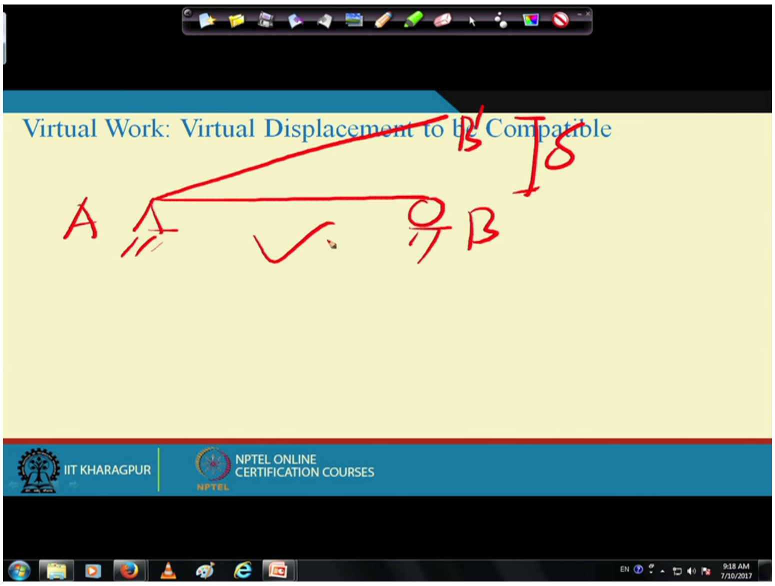

When I say compatibility it means the internal compatibility of the member should be satisfied. Then only we can say (displa) it is a valid virtual displacement, okay. Now let us understand the principle of virtual work, first for rigid body which is original proposed by Bernoulli. Now suppose this is a rigid body.

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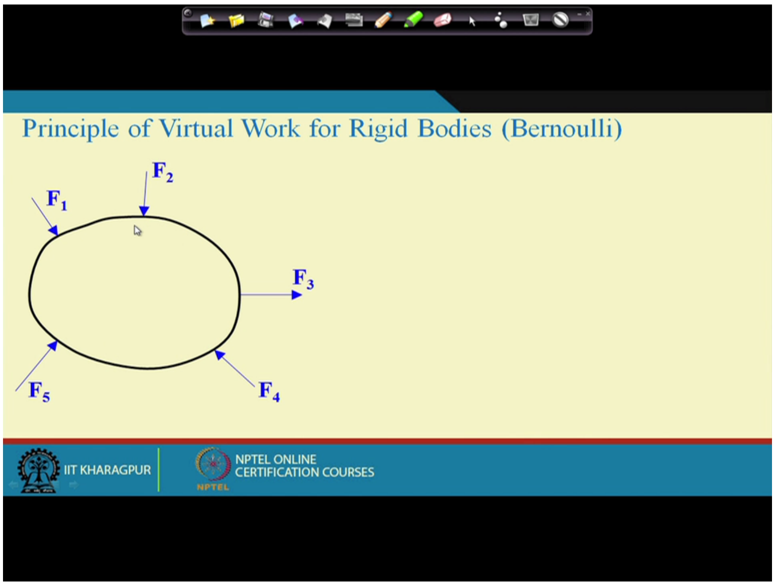

We know that in a rigid body now we have two forces. One is external force and internal force, okay. In a rigid body internal force is zero because the body moves as a rigid body. There is no strength develops in the body so internal forces are zero. And correspondingly internal work in the body is zero. So only work we have in the rigid body is the external work, the work due to the external applied load.

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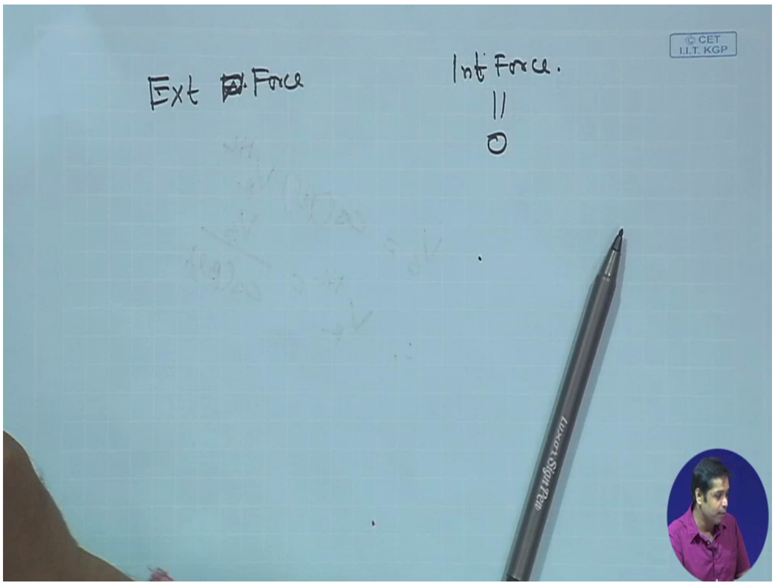

Suppose this is a rigid body and F1, F2, F3, F4, these are the forces acting on the rigid body, right? Now this is the centroid of the body. Now if the centroid of the body is C then FR is the resultant of all the forces and which is (ex) essentially summation of all the forces and their resultant will pass through this object if it is in equilibrium, okay.

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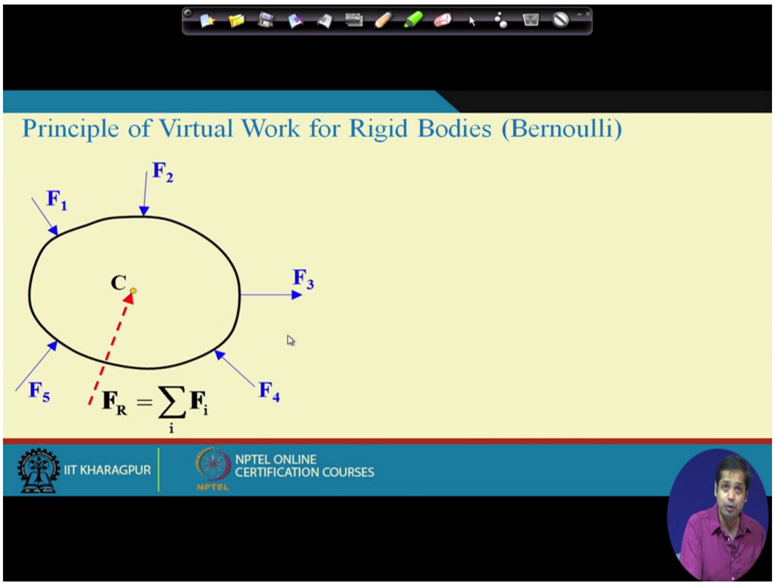

The resultant should be centroid of the body that is the equilibrium condition says, right? Now what we do is now instead of applying all the forces we can take same rigid body which is subjected to only the resultant forces. If the body is rigid body then we say that these two objects are equivalent system, right?

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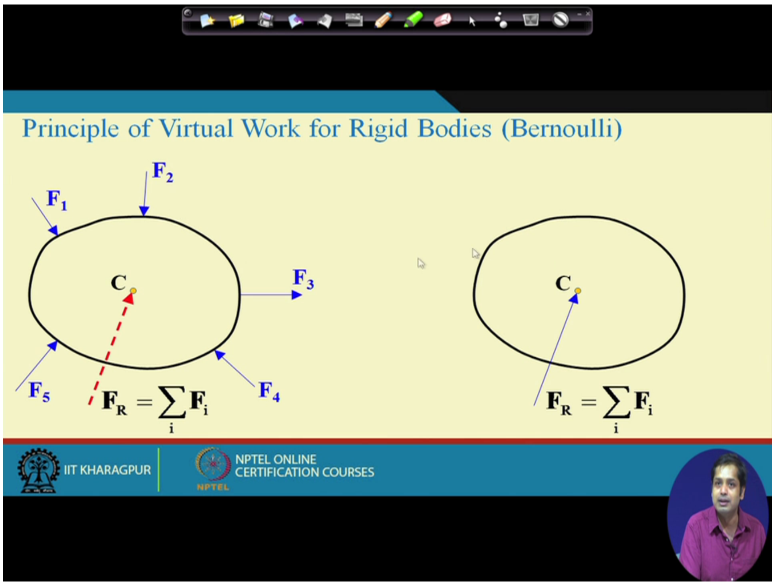

Now suppose we give some virtual displacement again the rigid virtual displacement in this body and suppose the body moves, the dotted lines are shown as the virtual displacement. Suppose the body, initially it was this and moves to this and the point C moves to C dash and C dash is the centroid of the same object, okay. The object shape, size, position of C dash with corresponding to the boundary, everything is same. It is just the translation of the body, okay.

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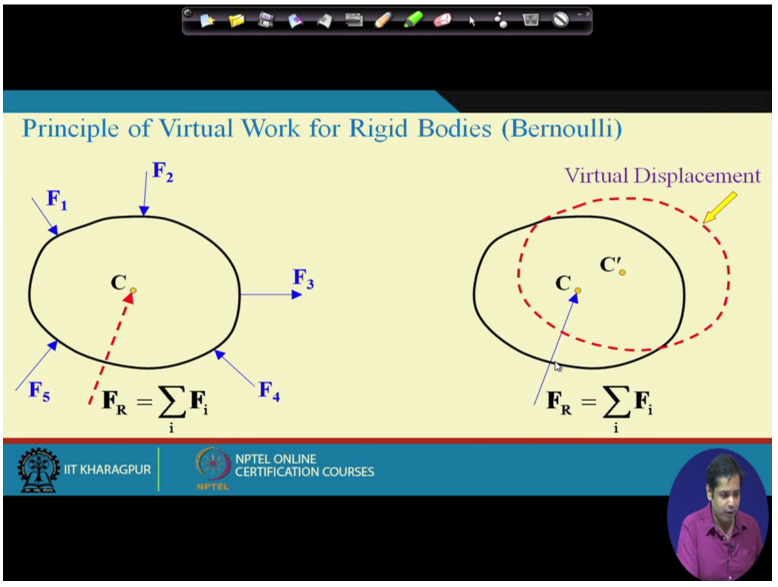

Now then what is the translation? Translation is rA. The movement of the body is C. The displacement is C, C dash which is rA. Now then what is the virtual work? Virtual work will be the force into the virtual displacement. This is the force and this is the virtual displacement.

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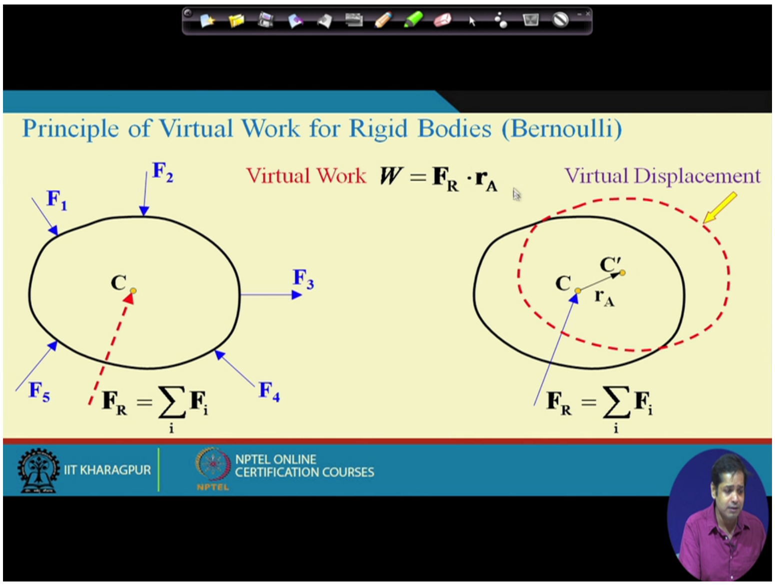

Now since the object is in equilibrium, now what is the value of FR? As we know equilibrium conditions says that summation of forces in any particular direction if we take they all are zero. So there is no net force and net moment acting on the body. So in equilibrium this is zero, okay. This is zero. These are all forces written in vector form. In equilibrium this is zero.

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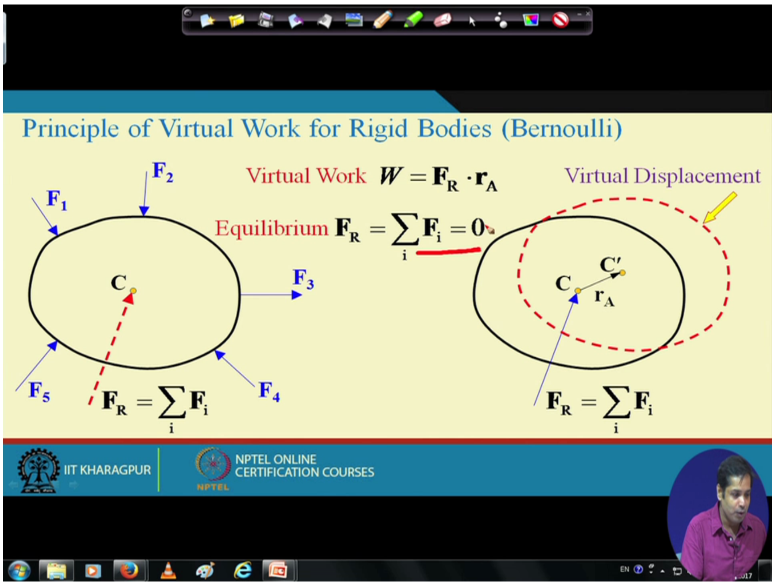

Now if this is zero if we substitute this here then what we have? We have W is equal to zero, right? So this is the principle of virtual work for rigid body. What then it says? It says that if a rigid body subjected to a system of forces is given a small virtual displacement then the total virtual work is equal to zero if the body is in a state of equilibrium. Now three things are important here and this is called principle of virtual displacement.

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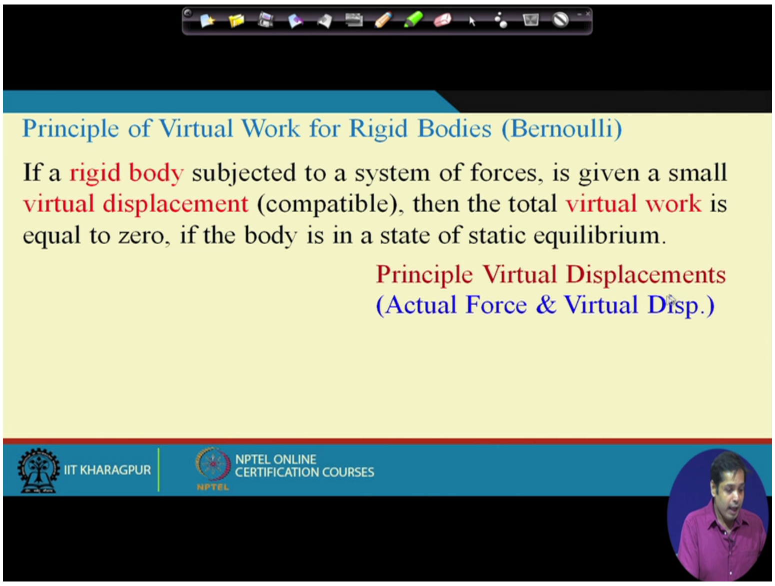

Now in this principle virtual displacement when we say work two things are involved, one is force, one is associated displacement. In principle of virtual displacement the force is actual force and the displacements are virtual displacement. Now the original principle virtual work as proposed by Bernoulli, there are three important things to be noted. One is it is for rigid body, okay.

But the kind of structure we will be talking about are not rigid. They undergo deformation, right? So we cannot use the principal exactly the way it is now. So this is for rigid body then another thing is, small is very important here. It is a small virtual displacement and then the total virtual work, okay.

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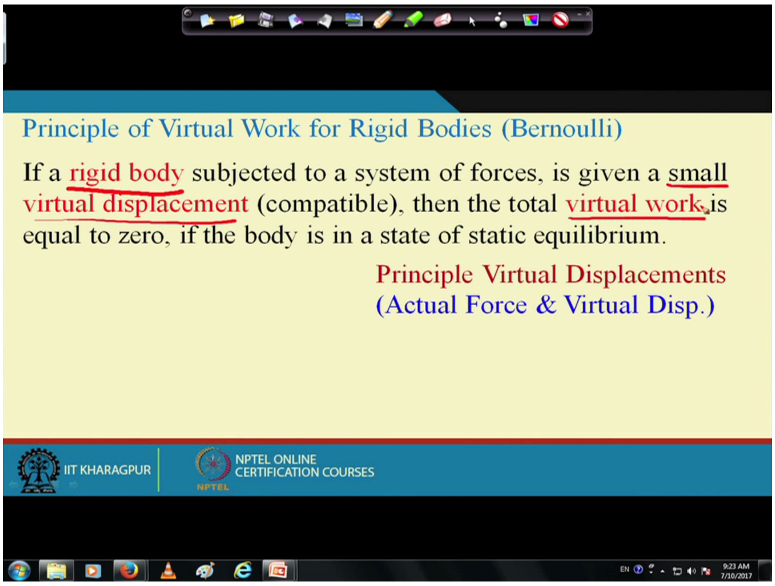

Now the three things are important. Now what we do is, this is very specific for the principle of virtual work for rigid body. What we need to do is, beyond this principle we need to generalize this principle so that the deformable bodies also can be included and whenever we talk about deformable body the internal forces are not zero, corresponding internal works are not zero. So we need to include internal work as well.

Now then there are three things we need to move beyond this. We need to generalize this principle and what we are going to include in this principle in the process of generalization, one is virtual force here, it is virtual displacement. Then similar to virtual displacement can we have virtual force. Means the displacement is real but then the object is subjected to some virtual force.

And we will see what is a virtual force and if we write the principle of virtual work in terms of virtual force it is written here in terms of virtual displacement that principle is called principle of virtual forces.

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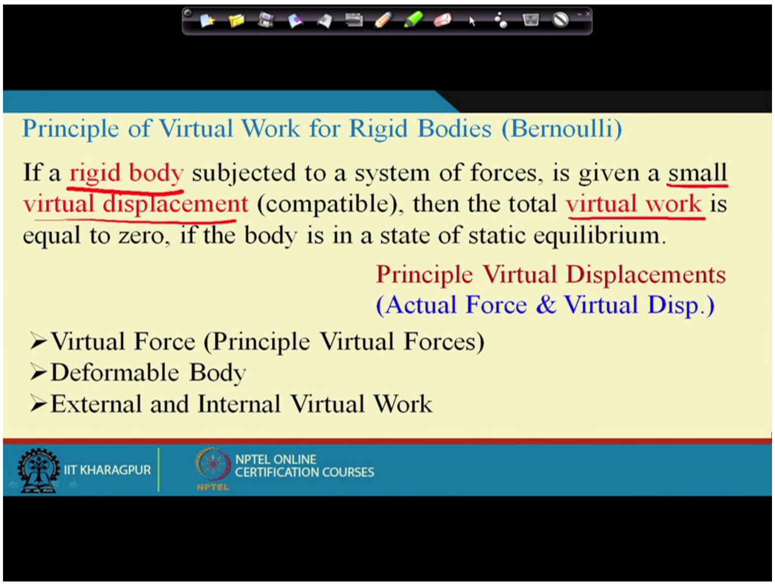

Like it is principle of virtual displacement. This is principle of virtual forces. Then of course consequently if the body is deformable then the external virtual work internal virtual work are not zero, okay. Now our objective is to generalize this principle of virtual displacement by considering.

But before we do that we need to understand what is virtual force? Then we need to understand what happens if the body is deformable? If the body is deformable then we need to include both external and internal forces and here it was very simple since there was no internal forces because there was a question of how external internal works are related. It was just only the work by the external forces.

Now when we talk about deformable body then we have external internal works both together and then how these external internal work are related to each other and that relation will give us a general principle of virtual work and the principle of virtual work. The possible displacement that we have just now seen is a very special case of that general principle. That we will discuss in the next lecture. So next lecture we will see what is principle of virtual forces? Okay. Thank you. See you in the next lecture.