Probability Methods in Civil Engineering

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Lecture – 20

Topic

**Probability Models using
Discrete Probability Distributions**

Hello and welcome to this lecture this in this module, we are discussing about common probability models and you know so far in the last couple of lectures we have discussed about different continuous probability distributions and we have seen their applications in different problems of civil engineering and in this lecture we will start the discrete probability distributions. So this discrete probability distributions that we have discussed earlier in some module we will just use those distributions and we will see their applications to handle different problems in for the different problems in civil engineering.
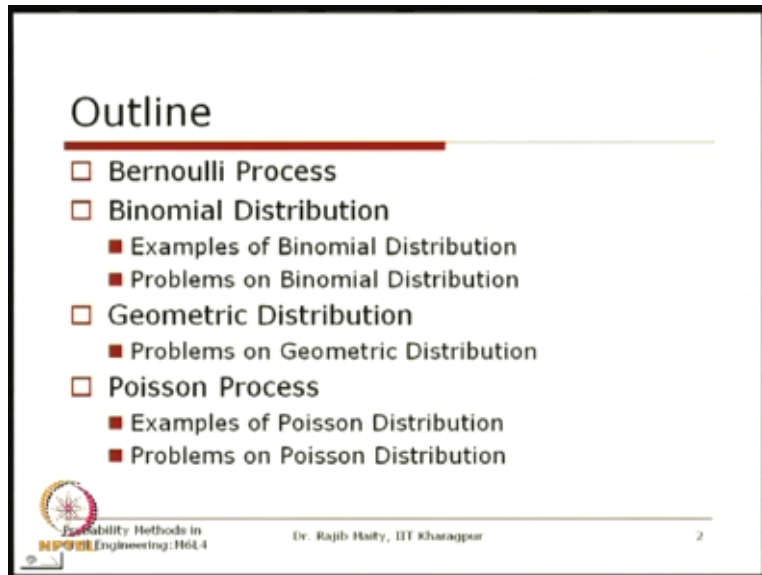
So our today's lecture we will deal with the probability models using discrete probability distributions and we will take couple of discrete probability distribution for example we will go.
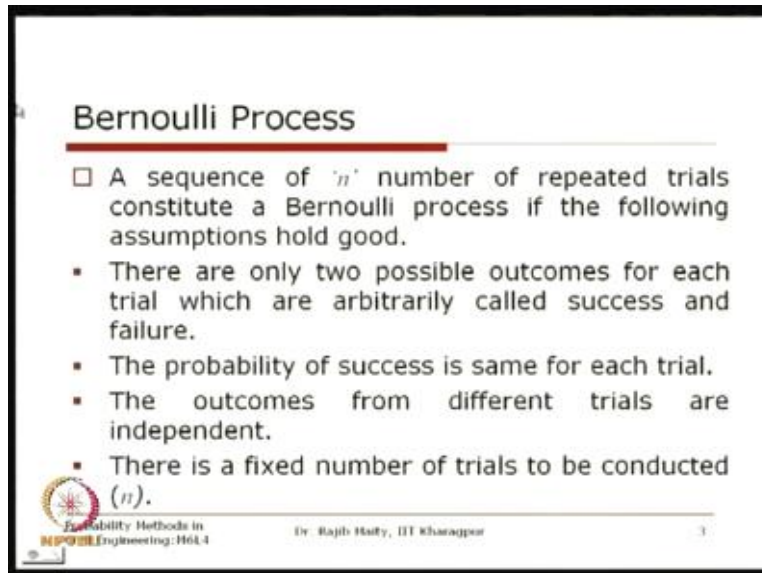
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Outline

☐ Bernoulli Process
☐ Binomial Distribution
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☐ Poisson Process
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First we will take that Bernoulli distribution, that binomial distribution, which is you know that there are certain process which we tell that it is the Bernoulli process there are based on some assumptions. So we will discuss those things first of all discuss that Bernoulli process and based on that, we will go to that binomial distribution even though this binomial distribution is quite simpler but simpler than other distribution that we have dealt with but so far as application in different problems is there it is really an important distribution for different civil engineering problem.

Particularly where the outcome is dichotomous means that there is only two possibilities are there so at that type of problem can be dealt with this binomial distribution then, we will go one after another that geometric distribution and we will discuss about it is basic properties and then we will take this distribution to solve some of the problems in civil engineering and similarly after that we will go to that Poisson process and the Poisson process is also, there are certain assumptions are there we will discuss that one and the Poisson distribution we will take to discuss its application to various to handle various problems in civil engineering. So we will start with this Bernoulli process first.

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This Bernoulli's process are based on some of these assumptions and these are actually you know that these properties and these assumptions are discussed in details in the earlier module when we are discussing about that common probability models so earlier methods in module 3 when we are discussing some of this standard probability distribution there we have discussed this thing in details here we will just quickly go through these properties and we will directly go to that problem which is basically the focus of this entire module.

So a sequence of n number of repeated trials constitute a Bernoulli process if the following assumptions hold good. The first is there are only two possible outcome for each trial which are arbitrarily called success and failure. So, as I was telling that outcome is dichotomous means, there are two possible outcomes are there. So and this outcomes can be arbitrarily called as a success and failure as also I have mentioned earlier this success and failure is nothing to do with their linguist meaning.

So any one of the outcome can be termed as success and another one can be failure even that tossing a coin you know that there are two possible outcome one is head and other one is tail

now if my goal is to get the head or that I am just going to means do some analysis that when I am going to get the first head or something like that.

That means that getting a head might be I can term as success and similarly that breaching that embankment of a canal then I can say that the breaching event is the success which obviously may not be the case from the social point of view. So these terminologies the success and failure in a Bernoulli process is completely arbitrarily can be assigned. The second assumption is the probability of success is same for each trial. So, the event that I am designate as success, that probability of that success should be same for each trial. So, for example for tossing a coin that getting a for a if the coin is fair then getting the head if it is I can term as success then getting the head probability is equals to 0.5

And that 0.5 will not change for whatever trial I go for third one is that outcome for from different trials are independent or this means that whatever I got earlier trial the outcome of this earlier trials should not have any influence to the outcome of this present trial so the individual trials are successive trials are independent to each other and the last there is a fixed number of trials to be conducted so this n so how many trials I am going to conduct that is known that is defined prior to the this process starts and that is designated as that n.

So based on this assumption now, if we want to want to designate one random variable that random variable is that if the probability of the success that is the occurrence of a particular event which we can call as success that in each trial is given by p as I was telling. So, this p will not change, then the probability of getting exactly x successful event among n trials in a Bernoulli sequence. Now we have to see here that this n trial this is predefined as was the fourth assumption of this Bernoulli's process and the random number that we are talking here is that the probability of getting exactly x successful events.

So this x now you can say that x can take any number any discrete number starting from 0 up to n so in an n trials I can get no success that means x = 0 or exactly 1 success, 2 success, similarly I can get all n can be the successful trial. So the random number that we are getting is the number

of success out of n trial. So, that random number is distributed through a probability it is a discrete probability function which is the binomial probability mass function.

And this binomial probability mass function is shown here that is probability of that X, exactly equals to x = n combination C. So this is a combination means that out of n events how many ways you can select x multiplied by p, which is the probability of success power x multiplied by $1 - p^{n-x}$ the power is n minus x. Now if we see here there are x success and the probability of success is p and there are so if there are x success out of n that means obviously there are n - x failure and if there are only two possible outcome and the probability of success is p and obviously probability of failure will be $1 - p$.

And as these are independent that is why we can multiply that their probability so this is that is why that this p^X multiplied by $1 - p^{n-x}$ this factor that is n choose x or n combination x, is this you know that this is n factorial divided by x factorial n - x which is this is also known as the binomial coefficient.

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Examples of Binomial Distribution

☐ Many real life problems in Civil engineering can be modeled through Binomial distribution:

- The water quality in a river on the downstream side of an industrial plant may or may not meet the pollution control standards on a particular day. Hence the daily water quality in such a river can be modeled through Binomial distribution.
- The maximum wind speed faced by a structure in a particular year may or may not exceed the design wind speed. Hence it may be modeled through Binomial distribution.

Now even though this distribution I was telling may be may look simpler compared to the other distribution that we have dealt with but it is having a very wide application to handle different problems in civil engineering particular whenever we see that there are two possible outcome of any particular even or any particular phenomenon, that we are trying to model there I can see if I can define that there are only 2 possibilities of the outcome then we will see that whether the other assumptions are satisfied of this Bernoulli's process or not then we can model that kind of problems by this binomial distribution.

So, here are two examples are given here say for example that the water quality in a river on a downstream side of an industrial plant may or may not meet the population that the pollution control standard on a particular day. Hence the daily water quality in such a river can be modeled through the binomial distribution. So, you see here that the random variable that the event that we are talking about is that may or may not meet the pollution control standard.

So we are not so in this type of problem we are not interested what is the level of this pollution which can be a continuous random variable but here we are just talking about the two events means it may meet the pollution controls standard or it may not meet the control pollution control standard. Now these are daily basis as it is written that on a particular day so these are daily basis so every day I will just see I will take the data either it is this has made the standard or this has not made the standard.

So there are two possible outcome and obviously these are independent from each other the other assumptions if we satisfy then this kind of problem can be handled through this binomial distribution. Similarly the maximum wind speed faced by a structure in a particular year may or may not exceed the design wind speed. Hence this also can be modeled through the binomial distributions.

So each and every structure or any structure when it designed there are several loads that we consider one is that wind load which is also one of the most important load while designing the structure. So that we assume some design wind speed for a particular that depends on various

thing its location then the type of the structure and etc. So that design speed once we decide and based on that we design the structure.

Now the question is that in a particular year whether the designs wind speed may or may not exceed. So each and every year one year I will take and I will take either of these two information whether it has exceeded at some event or not. So two outcomes are there and we can we can model this one also through the binomial distribution. Similarly you can we can we can think of many other cases.

That is after a rainstorm has occurred in a city certain critical junction whether that will be waterlogged or not. So there are two things it may waterlog or it may not waterlogged. So like that kind of thing on a highway stretch so whether there is an some accident or not. So that kind of thing. So here you remember that we are not interested about the quantity only we are interested that yes or no that kind of information. So that is the way the random variable is defined in Bernoulli's distribution or the binomial distribution.

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Examples of Binomial Distribution...contd.

- In a particular year, a region may or may not be water logged due to heavy flooding. Thus it may be modeled through a binomial distribution.
- Boulders may or may not be encountered during a soil boring. Hence it may be modeled through Binomial distribution.
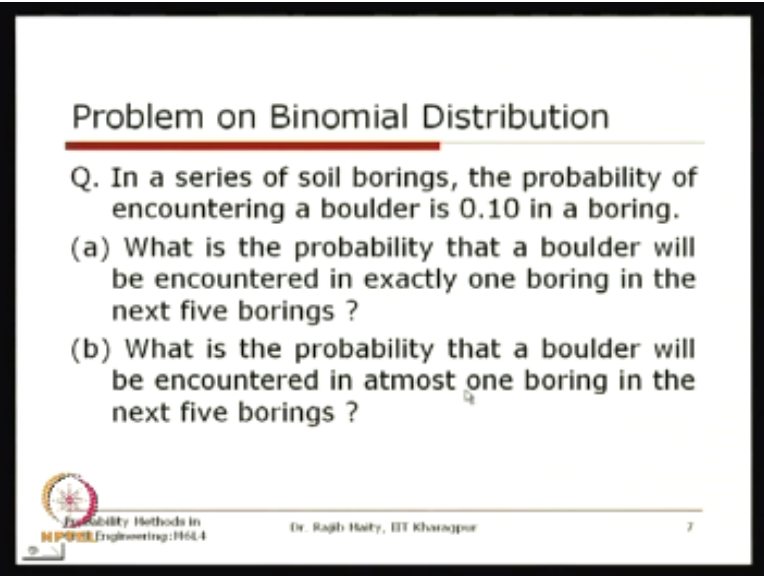
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So if we take one example that in a particular year a region may or may not be water logged due to the heavy flooding. So thus it may be modeled through the binomial distribution or through a geo-technical problem if we say that while we go for this soil boring through that boring there may or may not be some boulders can be encountered for a particular boring. So that boulders mayor may not be encountered during a soil boring.

So hence this also can be modeled through the binomial distribution. So we will take one of such problem some numerical problem to see these things.

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So on the problem states in a series of soil borings the probability of encountering a boulder is 0.1 in a boring. So here you see that this is the so there are two events we may encounter a boulder or we may not encounter a boulder. So we may encounter a boulder is if we arbitrarily term as success, then the probability of the success is known it was declared that success is 0.1 the probability of that success is 0.1.
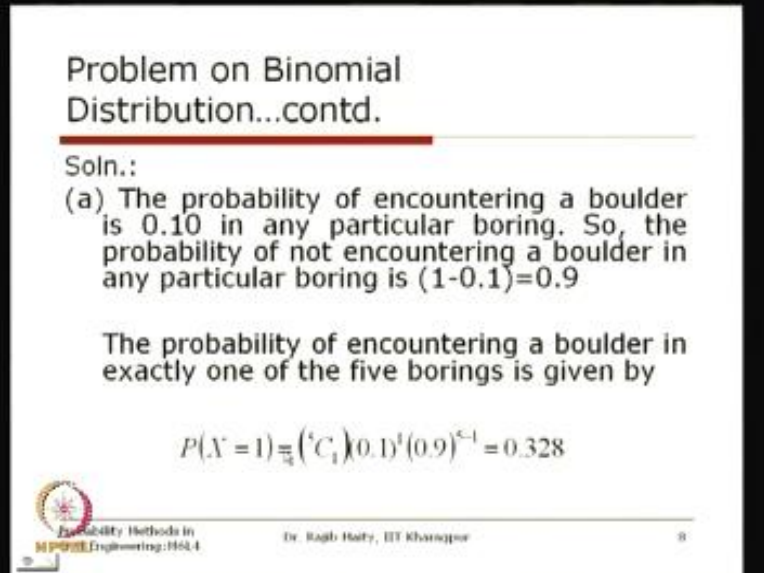
Now what is the probability that a boulder will be encountered in exactly 1 boring in the next 5 borings. So whatever the information given so number of so the fourth assumption of this

distribution is that of this process that Bernoulli process is that total number of trial should be known beforehand. So here it is written that the next 5 borings. So there are total 5 borings are there and we are looking for the probability that out of this 5 exactly 1 boring should encounter that boulder.

So what is the probability of that particular event? So, we know the probability of success we know the n, n = 5 here. So what is the probability to get that exactly 1 boring should encounter a boulder. So next version is, what is the probability that a boulder will be encountered in at most1 boring in the next 5 borings. So now the language is slightly changed one is that exactly 1 boring we are we are interested in the previous question and now here the utmost 1 boring. So utmost 1 boring means it may be 0 or it may be 1.

So we have to just sum up these two probabilities one is for the 0 encounter means a case number of cases to encounter a boulder is 0 and number of cases encountered a boulder is 1. So if we add these two then we will get the thing utmost one boring in the next 5 borings.

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Problem on Binomial
Distribution...contd.

Soln.:
(a) The probability of encountering a boulder
is 0.10 in any particular boring. So, the
probability of not encountering a boulder in
any particular boring is (1-0.1)=0.9

The probability of encountering a boulder in
exactly one of the five borings is given by

$$P(X=1) = \left({}^5C_1\right)(0.1)^1(0.9)^{5-1} = 0.328$$

So if we solve this one that the probability of encountering a boulder is 0.1 in any particular boring. So the probability of not encountering a boulder in that particular boring because you know this is the two possibilities are there so the other probability or the probability of failure is the 1-0.1which is 0.9. Now the probability of encountering a boulder in exactly 1 of the 5 borings is given by P probability of X = 1 = 5C1. This is a nCx that we use there 5C1, 0.1^1 and 0.9^{5-1} which if we calculate we will get that 0.328. So, exactly 1 boring will face will encounter 1 boulder is equals to 0.328.

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Problem on Binomial Distribution...contd.

(b) The probability of encountering a boulder in at most one of the five borings is given by

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$
$$= \left({}^5C_0\right)(0.1)^0(0.9)^5 + \left({}^5C_1\right)(0.1)^1(0.9)^4$$
$$= 0.590 + 0.328 = 0.918$$

And the second was that probability of getting the at the most 1 out of this 5 borings is that. So, we have to add two such events that X = 0 and probability X = 1 and X =1, we have just seen it is 0.328 and probability of facing 0 is 0.59. So 5C0. So nC now x is here $0,0.1^0$ and 0.9^5 . So if we get this one this is 0.918 the probability of encountering a boulder utmost 1 of these 5 borings.

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So another example on this binomial distribution is that a dam was built to a certain height above the mean water level in the river and has a probability of 0.1 of being over topped. Now here you see that whether the water level of the dam will be such that the water will overtop and for it has a probability. So that we are now terming as a success is that whether it is overtopped or not, is that has a probability of 0.1.

Then what is the probability that dam will be overtop within the return period and second is that if the dam is overtopped then the probability of damage is 80 %,what is the probability that dam will be damaged within 3 years.

Problem on Binomial Distribution...contd.

Soln.:

i) The return period of the event of overtopping is $\bar{T} = \dfrac{1}{p} = \dfrac{1}{0.1} = 10\ years$

The probability of overtopping in 10 years

$$= 1 - (1 - 0.1)^{10} = 1 - 0.3487 = 0.6513$$

Now to solve this problem, as we have seen that the return period of the event of over topping is T which is the inverse of the probability of that event that is overtopping that is 1/ 0.1 that is 10 years. So now the question is the probability of overtopping in 10 years. So if so the 1 minus if you see this one that 1 - 0.1 which is 0.9 which is the probability of not overtopping. Now this probability power 10 is gives that it will not be overtopped in 10 successive years.

Now if we want to know that what is the probability of overtopping then this probabilities should be subtracted from 1. So by calculating we get that the probability comes to be 0.6513 which is the probability of overtopping in10 years that means the return period, so this is first part.

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Problem on Binomial Distribution...contd.

ii) For the dam to be damaged in 3 years, the dam may be overtopped 0,1,2 or 3 times in the 3 year period.
So the probability of no damage in 3 years

$= 1(0.9)^3 + (0.2)[3(0.1)(0.9)^2] + (0.2)^2[3(0.1)^2(0.9)] + (0.2)^3(0.1)^3$

$= 0.7738$

So the probability of damage in 3 years

$(1 - 0.7738) = 0.2262$

And the second part which is the probability of damage in 3 years. Now we will do this problem again that we first will find what is the probability of no damage in 3 years. Now if we see that how many years that it can overtop first that is in 3 years it may so happen that none of these 3 years is overtopped. So the case is 0 it may so happen that only 1 year is overtopped or 2 years out of these 3 years is overtopped or it may be all 3 years.

In all 3 years it can be overtopped. Now it is given that if it is overtopped then the probability of damage is 0.8. So the probability of no damage in 3 years if we want to calculate then we have to calculate that for all these 4 cases. Now if we take that the first one that is when it is overtopped for 0 cases. Now if that is it is not at all overtopped in 3 years that means the probability of no damage should be equals to 1 because it is not over topped so there should there is no question of over topped so that is why that probability is 1 now if you just want to expand this one then you will see this particular case that is you know that.

C that is n C how many times the success is now here a n = 3years and we are saying that it is for 0 years it is overtopped and then the probability of success is power 0 and this $1 - p$ power3 that is $3 - 0$ basically now you know this p is equals to your 0.1and so this is 0.9^3 and this is obviously1 and this should be multiplied with that chance of damage and you know that if it is not over topped the probability is equals to 1 so which is basically your 0.1 power 0 multiplied by 0.9^3 which is so1 0.9^3 so this is the first entry that you can see.

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Here in this expression and the second one is that this 3 is coming from that 3 C1 then probability of success which is 0.1^1 and 1 - 0.1 that is 0.9^2 and this is the case when only 1year out of these 3 years the over topped has happened so and that probability of no damage is again1 - 0.8 which is 0.2 so that is multiplied here now if it is the 2 years out of this 3 whether it is overtopped or not and for that it is no damage will be the probability of no damage will be 0.2 square and which is again that this 3 is again that 3C2 0.1^2 and 0.9 p^{3-2} that is 1and similarly the last one is the case.

For when all 3 years it is overtopped now if we calculate this one we will get 0.7738 which is the probability of no damage in 3 years now our question is to get that what is the probability of damage in 3 years which is subtracted from 1 = .2262.

Geometric Distribution

If the probability of occurrence of an event in any particular trial is p, then the probability that the first occurrence of the event is on the tth trial in a Bernoulli sequence is given by

$$P(T = t) = p(1-p)^{t-1} \quad t = 1, 2, \dots$$

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Okay next we will go to this geometric distribution we will first know that what is the difference between this binomial distribution and this geometric distribution because the both the distribution basically is a from the same Bernoulli's sequence now here in this geometric distribution it says that if the probability of occurrence of an event in any particular trial p then the probability that the first occurrence of the event is on the t-eth trial in a Bernoulli sequence is given by the probability of T = t p x 1 – p^{t-1} so what we are looking for what is the random variable that we are considering in this geometric distribution.

Is that how many trials you need to get the first success so success again is defined arbitrarily that is the how many trials that we should go so here you can see that this probability of success and this out so this is power1 and 1 - p this is the probability of failure in t- 1 event so there are total t events are there out of which 1 that 1 is success the last one and this 1 - p before that t - 1 cases this is a failure case.

So that we are interested to know the distribution of this T that how many trails you need to get the first successful event so this random variable is modeled through this geometric distribution.

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Geometric Distribution...contd.

☐ Mean Recurrence Time or Return Period

In a Bernoulli sequence, the average time between two successive occurrences of an event is called the Mean Recurrence Time or Return Period.

The Return Time 'T' is expressed by Geometric Distribution and is given by

$$\bar{T} = E(T) = \sum_{t=1}^{\infty} t \, p(1-p)^{t-1} = p\left(1 + 2(1-p) + 3(1-p)^2 + \ldots\right)$$

$$= \frac{p}{[1-(1-p)]^2} = \frac{1}{p}$$

Now this mean recurrence time or this return period that is in a Bernoulli sequence the average time between two successive occurrence of an event is called the mean recurrence time or return period the return periods or return period T this should be period the return period T is expressed by geometric distribution and is given by the T bar that is expectation of this T = T = 1 to ∞ t multiplied by its probability this you know that how we get this expectation we have discussed in earlier modules that is if we get this one.

Then we will get this infinite series as this the p is taken common and this 1 + 2 x 1 - p + 3 x 1 − p² and this way and this is the expression of this infinite series is that p by 1 - 1 - p whole square so this is 1 by p so the mean recurrence time of that event that how frequent that particular event particular event means here we are talking about that the arbitrarily selected successfully event how frequently that is occurring is nothing but is that 1 by p it is probability so this is how that this recurrence time and the T return period and the probability is related to.

We will take one problem on this geometric distribution that a multistoried building has been designed against a 30 year earthquake what is the probability that the design earthquake intensity will be exceeded for the first time on the third year after the structure is built so here that 30 years earthquake means this is the event that we are talking about that the event of the earthquake has a return period of 30 years so the probability of success again that having the earthquake is the term as a success.

So that the probability of the success we should get from its return period like the way that we have discussed just now and the question is what is the probability that the design earthquake intensity will be exceeded for the first time on the third year after this after the structure is built so this is that the first event will occur at the third year itself so that we can model through this geometric distribution so the first success after some particular time stamp and here it is the third year second question is.

What is the probability that the first such earthquake intensity will occur within 3 years after the structure is built so this is that on the third year and this is within the 3 years itself that design

thing that the design earthquake will occur and which is we have designed for the 30 years earthquake.

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Problem on Geometric Distribution...contd.

Soln.:

(a) The probability of occurrence of a 30 year earthquake in any given year is 1/30=0.033

The probability that the design earthquake intensity will be exceeded for the first time on the third year after the structure is built is given by

$$P[T = 3] = (0.033)(1 - 0.033)^2$$

$$= 0.0308$$

So to solve this one first of all we have to get what is the what is the probability of getting such event and as it is shown in the problem that the probability of occurrence of a 30 year earthquake so the return period is 30 years here so that probability of occurrence of such event should be 1 by 30 so which is 0.033 so obviously it is up to 3 decimal is considered so the probability that the design earthquake intensity will be exceeded for the first time on the third year after the structure is built is given by that T equals to the probability that T = 3 is p multiplied by 1 - p n -1.

So this n - 1 here is that 3- 1 so it is 2 if we get this one we get the probability equals to 0.0308. so this is the probability that the earthquake will occur at the third year so you can see that this is quite expected that when we design the structure for 30 years return period so that earthquake will occur at the third year itself so one tenth of the total design total design period so this obviously the probability will be very less and we got the probability 0.0308.

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Problem on Geometric Distribution...contd.

(b) The probability that the first such earthquake intensity will occur within 3 years after the structure is built is given by

$$P[T \le 3] = \sum_{t=1}^{3} (0.033)(1-0.033)^{t-1}$$

$$= (0.033)(0.967)^{1-1} + (0.033)(0.967)^{2-1} + (0.033)(0.967)^{3-1}$$

$$= 0.0333 + 0.0319 + 0.0308$$

$$= 0.0960$$

In the second was that the probability that the first such earthquake intensity will occur within 3 years after the structure is built now here within the 3 years after the structure is built means it can happen any time between that it may occur in the first year it may occur in the second year and third year and we have seen that in the Bernoulli's process that successive events are independent so what we have to do is that we have to get this probability for what is the probability for the event to be occur in the first year what is the probability that event may occur in the second year.

And what is the probability that event may occur in the third year so these 3 probabilities we should add up so if this is exactly what is done so probability T is less than equals to 3 is that T equals to 1 to 3 and this distribution so 0.0331 - 0.033 power t minus 1 so we can put this one as this t equals to 1 first so we will get this expression t equals to 2 and t equals to 3.

So occurring and the probability of the occurrence of the event in the first year itself is 0.0333and then it is. 0.0319 and 0.0308 for the third year. So, if we add up this probabilities, we get the 0.0960. So, the probability that the first such earthquake intensity will occur within the 3

years, we are using the word within the, within the 3 years after the structure is built is 0.0960 which is slightly obviously this will be slightly more than that what we got in the last one.
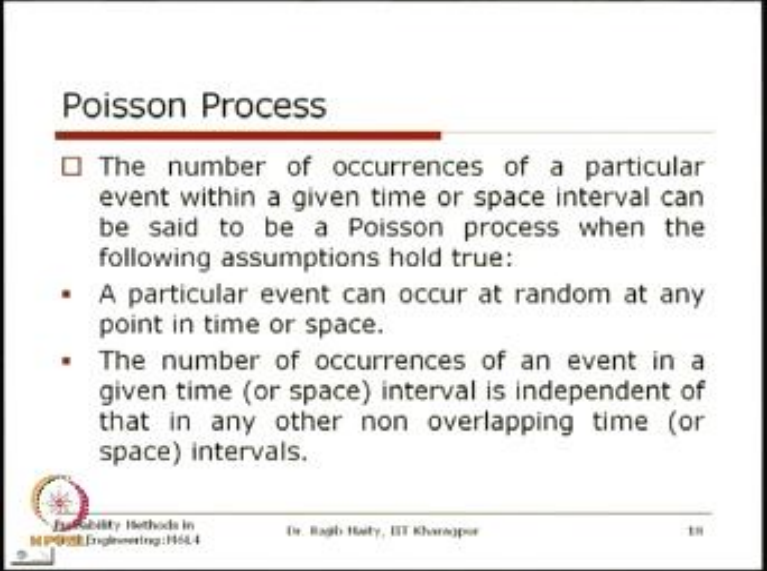
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Problem on Geometric Distribution...contd.

Soln.:

(a) The probability of occurrence of a 30 year earthquake in any given year is 1/30=0.033

The probability that the design earthquake intensity will be exceeded for the first time on the third year after the structure is built is given by

$$P[T = 3] = (0.033)(1 - 0.033)^2$$
$$= 0.0308$$

Which is exactly at the third year is this 0.0308.

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Problem on Geometric Distribution...contd.

(b) The probability that the first such earthquake intensity will occur within 3 years after the structure is built is given by

$$P[T \le 3] = \sum_{i=1}^{3} (0.033)(1-0.033)^{i-1}$$

$$= (0.033)(0.967)^{1-1} + (0.033)(0.967)^{2-1} + (0.033)(0.967)^{3-1}$$

$$= 0.0333 + 0.0319 + 0.0308$$

$$= 0.0960$$

Because we are including the other years also that is first one and second one. But obviously, if we see that there are the written period is 30 years, so, that 30 years earthquake intensity is considered in the design. So, both this probabilities will be very less and that is what we got that second case which is 0.0960.

Okay, next we will discuss about other one that is the Poisson process. So, this is also based on some of the assumption like similar to that Bernoulli's process. So, here the number of occurrences of a particular event within a given time or space interval can be said to be a Poisson process when the following assumptions hold true. Now, we can say here is that, now the specified duration of the time or obviously, over the space or length of the highway and on so, or time is in the temporal directions and number of events over particular time over a month, over a year like that.

So, now, that duration the temporal gap or the space limit over which, that is defined and what we are interested to know is that, how many such events can occur in this, over this duration or over this special limits. And the assumptions are a particular event can occur at random at any point in time or space. So, it has no dependence of this, what has been occurred earlier. So, if I just consider in the temporal direction so over suppose that we are considering a particular time from 0 to T, so, it can that particular event can occur any time within this time frame and the number of occurrences of an event in a given time or space interval is independent of that in any other non overlapping time or in intervals.

So, that is what means what he has occurred earlier, it should be independent of what is going to occur in the next time step. So, if we consider to suppose that from the total time span, that we are talking about this 0 to T and in between that if I just segregate this total time into two parts and this two parts are non overlapping to each other, then the number of occurrence within the first interval and the second interval should be independent to each other.

So, basically there is no memory, if the process does not have any memory what has occurred in the last time strip or that last duration of the time.

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The probability of occurrence of an event in a small interval Δt is given by the $\lambda \Delta t$. So, now, if we just consider a small time interval Δt over the time that we are considering and obviously, this can be the in the special direction also we can say we can denote that one as a Δs. So, over the small interval of time or the space, that is given by the probability, that it will occur in that time interval is the $\lambda \Delta t$, where this λ is the mean rate of occurrence of that event. So, this λ is basically now, how we will get is the average that number of occurrence of that time, number of occurrence of that event per unit time, basically that is what is shown by this parameter λ.

So, this lambda is known, before you can define a Poisson process. The probability of more than one occurrence of an event in the small interval of Δt is negligible. So, in that small time, so this is basically the infinite small so in this time in interval, there is the possibility of occurrence of more than one event is negligible and that is based on what we are just even that this a mean rate of occurrence of that event, which is denoted by λ.

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So, as per the assumptions of the Poisson process, the number of occurrences of an event X_t in the time t is given by the Poisson process. So, here the random number that we are talking about is that number of occurrences of an event X_t so over the total time from 0 to that particular time interval t so over this interval, what is the total number of occurrence, that we are interested here. Where this you can see that, this t can, t is a continuous variable so, t can have a continuous length. So, at any point of time, it can happen. So, over this continuous time frame, over this continuous time or it can be space also, over this continuous special extent how many times that particular event can occur.
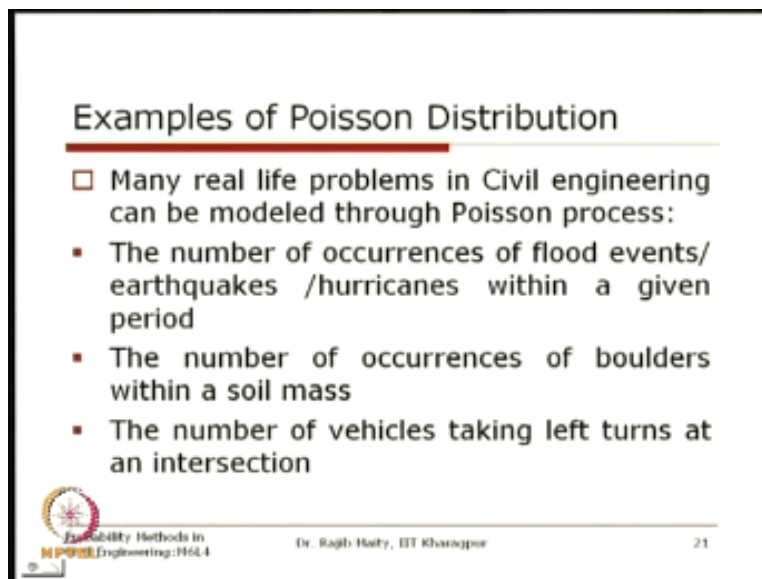
So, when we are looking for this binomial distribution, there are some discrete trials over which how many times that event can occur and here in the Poisson process over a continuous time

frame or continuous special extent how many times that event can occur. So, this is the random variable of which distribution is given through this one that probability of this is the variable on which we are expressing, this distribution is x and the parameters is the is the λt, here you can see that if the λ is the mean rate of occurrence, then this λt is a total number of expected that it can occur within that time interval of t.

So, that X_t is exactly equals to x is that $(\lambda t)^x / x! \, e^{-\lambda t}$ now where this parameter, there is λ is greater than 0, t is greater than 0 and x can take any value from 0, 1, 2 and up to ∞. So, here the support of this x, you can see that, it can start from 0 and go up to ∞, all are discrete the number of occurrences. But in the earlier that binomial distribution, it was, it can take from the 0 to that number of total number of trials considered that is up to n.

So, here that what I told that this λ is the mean rate of occurrence that is average number of occurrence per unit time.

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Now, many real life problems in civil engineering can be modeled through this Poisson process, first is that number of occurrences of the flood events, earthquake, hurricanes within a given

period. Now, when so in this Poisson distribution, when we are talking about we are just talking about some discrete variable which is the number of occurrences, but that total time frame that we define should be a continuous one, so over a given period. So, basically this number of occurrences can happen from 0 to ∞, because this the time frame or the special extent we are considering as a continuous medium.

The number of occurrences of the boulders within the soil mass. So, for a particular soil boring, so how many boulders can be encountered? So, that is so, in a soil boring means it is from the surface to certain depth. So, in this special direction at any point, at any depth that boulder can occur and any numbers of boulder can happen so, this number of occurrences of boulders within the soil mass or in the within a particular depth of soil boring that also can be modeled through this Poisson distribution.

The number of vehicles taking left turn at an intersection can also be modeled through this Poisson distribution over a particular, obviously, over a particular time. So, suppose that in one hour, this many vehicles will take the left turn at a particular intersection. So, that is can also be modeled through the Poisson distribution.

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Now, if we take one example of this Poisson distribution, the historical records of rainstorm in a town indicate that in a month of May, the average number of rainstorm is 3 for the last 30 years. So, we are considering a particular month of the year and in that month, there is an average number of rainstorm for the town is 3, which is observed for this last 30 years. So this we can say that what is the average occurrence of the event, assuming that the occurrence of the rainstorm is a Poisson process what is the probability that there would be no rainstorm in the month of May in the next year.

So that May, that is the total duration of this time, anytime the rainstorm can occur and we have seen that over that period it is there are 3 rainstorm is the is the value that we have seen from the historical record and the question is that what is the probability that there will be no rainstorm in the next year. Other two questions are, what is the probability that there would be exactly 3 rainstorms in the month of May, in the next year, what is the probability that there will be 2 or more rainstorm in the month of May in the next year.

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Problem on Poisson Distribution...contd.

(b) what is the probability that there would be exactly 3 rainstorms in the month of May in the next year?

(c) what is the probability that there would be 2 or more rainstorms in the month of May in the next year?

So basically the first question is we have to say that exactly 0 rainstorm, second one it is exactly 3 rainstorm and third one is the so that summation of the probability of having 2 or 3 or 4 or 5 or any numbers can go to infinity. Now so to solve this third one what we can show is that what we can do is that the total probability is 1and from that 1 we can deduct that what is the probability of the occurrence of the rainstorm.

No rainstorm that is 0 event one, we can if we just sum up these two probability in case of 0 rainstorm and 1 rainstorm sum up and if we deduct from the 1 then we will get what is the probability, that it will occur 2 or more rainstorms. So without solving you can say that the probability of this having this0 rainfall and this exactly 3 rainfall will be much lesser compared to this last one where we are basically considering all the possibilities except that 0 and 1 case.

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Problem on Poisson Distribution...contd.

Soln.:

(a) The average number of rainstorms in May is 3, i.e, the mean rate of occurrence $\lambda = 3$

Now, $\lambda t = 3(1) = 3$

The required number of occurrences $x = 0$.

The probability that there would be no rainstorms in the month of May in a particular year is given by

$$P(X_t = 0) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} = \frac{3^0}{0!} e^{-3} = 0.0497$$

So here you can see that the average number of the rainstorm in the month of May is 3,that is the new rate of occurrence the $\lambda = 3$ and now this λ t, when we are considering here this is a 1 month and there is average occurrence is 3. So this λ t = 3 and the required number of occurrences here or the probability, that we are looking for the number here is x = 0 for the first case. The

probability that there will be no rainstorm in the month of May in a particular year is given by probability X t = 0 λt^x /x, e^λt where this λt, we know as the 3 and this x = 0 λ t is 3 here it is shown. The probability that there will be no rainstorm that is number of rainstorm is 0 is equals 0.0497.

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Problem on Poisson Distribution...contd.

(b) The required number of occurrences x =3.

The probability that there would be exactly 3 rainstorms in the month of May in a particular year is given by

$$P(X_t = 3) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} = \frac{3^3}{3!} e^{-3} = 0.224$$

Second the required number of occurrences x = 3 here. So, number of occurrences of rainstorm is exactly equal to 3. So, we have to just put that in place of 0 we will just put 3 now in this expression and we can see that this number of occurrences here is the 0.224, which is obviously because this is the expected number of occurrences. So, this have the higher probability, then compare to the earlier case, where we have consider the 0 of occurrences.

Problem on Poisson Distribution...contd.

(c) The required number of occurrences $x = 2$ or more.

The probability that there would be 2 or more rainstorms in the month of May in a particular year is given by

$$P(X, \geq 3) = \sum_{x=2}^{\infty}\left(\frac{3^x}{x!}e^{-3}\right)$$

$$= 1 - \sum_{x=0}^{1}\left(\frac{3^x}{x!}e^{-3}\right)$$

$$= 1 - 0.0497 - 0.0248 = 0.9255$$

Now the third one is that the required number of occurrences is the 2 or more. So that the probability for the occurrences of the rainstorm occurrences x equals to 2 or more means, that includes that x = 2, 3,4, 5 like this up to infinity. So the probability that there will be a 2 or more rainstorm in the month of May in a particular year can be given by basically from x equals to 2 to infinity is, put this 3 that is the $\lambda t^x e^{-\lambda t}$.

So this, what I was telling that this we can calculate just by 1 - x =0 to 1 of this one. So the total probability, so this from 0 to infinity this is total is obviously equals to 1 from the basic principal of this probability mass function the total probability should be equals to1.So this 1 minus, now we can put this 0 that is we have already got this one that for x - 0, the probability is 0.0497 and similarly for x equals to 1, it will come0.0248.

So that the probability that there will be 2 or more rainstorm in the month of May is 0.9255, so you can say that this is basically taking care of almost full support of this random variables, so which is much higher than compare to the other 2 cases where it is 0.9255.

Problem on Poisson Distribution...contd.

Q. Along the highway, the mean number of petrol pumps is 4 per 100 kms. Assuming that the location of petrol pumps along the highway is a Poisson process,

(a) What is the probability that there would be exactly 1 petrol pump within the next 25 kms ?

(b) what is the probability that there would be atmost 3 petrol pumps within the next 100 kms ?

Now another problem on this Poisson distribution is that along a high way, the mean number of petrol pumps is 4 per 100 kilometers. So, this is now in the special direction earlier it was in the temporal direction now this example is on the special direction. So over the length of 100 kilometer, on an average there are 4 petrol pumps are there. Assuming that the location of the petrol pumps along the high way is a Poisson process means, assuming that it is Poisson process means all the assumptions that we have shown earlier should be followed.

So if we can assume if those assumptions are satisfied then the question is what is the probability that there would be exactly1 petrol pump within the next 25 kilometer and second what is the probability that there would be at most 3 petrol pumps within the next 100 kilometer.

Problem on Poisson Distribution...contd.

Soln.:

(a) The required number of occurrences $x = 1$ in 25 kms.

The average number of petrol pumps is 4 per 100 kms, i.e, the mean rate of occurrence $\lambda = 4/100$

Now, $\lambda t = \left(\dfrac{4}{100}\right)(25) = 1$

The probability that there would be exactly 1 petrol pump within the next 25 kms is given by

$$P(X_t = 1) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} = \frac{1^1}{1!} e^{-1} = 0.368$$

So the first one the required number so required number of occurrence here is that x = 1 in 25 kilometer. And the average number of petrol pumps is a 4 per 100 kilometer. So in 100 kilometer there are forces so the mean rate of occurrences of 1 petrol pump is the 4 by 100. So if this one is so 1/ 25 is the value of this λ, now that λ t. So here means even though the notation, we are using exactly similar to this distribution 1, so here the t means that in the special direction.

So we are interested now in this 25 kilometer. So 25kilometer, so this t is here 25 and this $\lambda = 4$ petrol pump per 100kilometer. So it is 4 /100. So, which is the λt is given the value equals to1.So the probability that there would be exactly1 petrol pump within the next 25 kilometer that means, the X t =1 with the λt value is again equals to 1. So λt^x/x factorial e$^{-\lambda t}$ that means, λt is 1, 1 power exactly 1 petrol pumps 1 power 1 factorial e^{-1}.

So the probability that we get is 0.368 that is within the next 25kilometer the probability of getting 1 petrol pump is 0.368.

Problem on Poisson Distribution...contd.

(b) The required number of occurrences is atmost 3 in 100 kms.

The average number of gas stations is 4 per 100 kms, i.e, the mean rate of occurrence $\lambda = 4/100$

$$\text{Now, } \lambda t = \left(\frac{4}{100}\right)(100) = 4$$

And second is that the required number of occurrences is at most 3 in 100 kilometer so what is the probability that there will be at the most 3 petrol pumps in the next100 kilometer. So the average number here again is this 4 by that 100. So the λ is obviously 4 /100, now if we calculate the λt in this case now, the t here is your 100 kilometer that we are considering, λ is same 4 / 100. So, here the $\lambda t = 4$.

Problem on Poisson Distribution...contd.

The probability that there would be atmost 3 petrol pumps in the next 100 kms is given by

$$P(X_t \leq 3) = \sum_{x=0}^{3} \left[\frac{(\lambda t)^x}{x!} e^{-\lambda t} \right]$$

$$= \left(\frac{4^0}{0!} e^{-4} \right) + \left(\frac{4^1}{1!} e^{-4} \right) + \left(\frac{4^2}{2!} e^{-4} \right) + \left(\frac{4^3}{3!} e^{-4} \right)$$

$$= 0.0183 + 0.0732 + 0.1464 + 0.1952$$

$$= 0.4331$$

Now if we do this one that the probability that there would be at most 3 petrol pumps in next 100 kilometer is given by that X t is less than equals to 3. So it can happen that, there is no petrol pump; there is only1 petrol pump, 2 petrol pumps or 3 petrol pumps. So, we have to sum up these probabilities, stating that x equals to 0 to 3 and this probability with the λt value here as we have seen in the last slide that $\lambda t=4$.
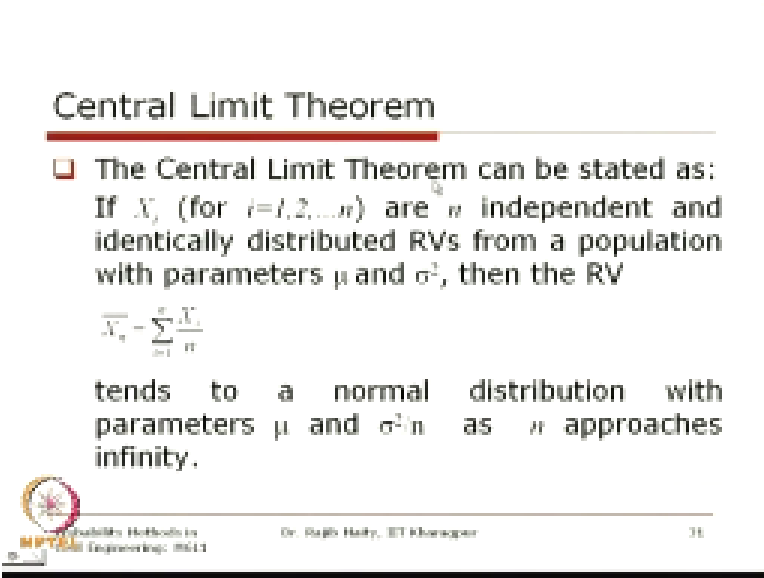
So this 4 first case is equals to 0. So, 4^0 / 0 factorial equal e^{-4} similarly 4^1 then 4^2, 4^3 this probabilities, if we just get the first one is 0.0183, 0.0732, 0.1464, 0.1952. So, if we add up these things then we get that0.4331. So the probability that there will be at most 3 petrol pumps in next 100 kilometer is 0.4331. Now when we are so this type of problem that we have discussed so far, both in this Bernoulli's distribution and for this Poisson distribution, geometry distribution.

So there are those parameters, those are the probability of the success is generally obtained from the historical record. Now this historical record, when we are talking about basically the problems, that we are discussing may look that, if this information is supplied, then I can get this probabilities, but when we are taking that real life example, then maybe those probabilities we have to obtain from the whatever the data is available to us.

For example, the one example we have shown that out of this last 30 years, it has been seen that there are 3 rainstorms in the month of May. So, that last 30 years record, we can just see and that may be in the first year there are 6 rainstorms, second year there is 0, third year there is 4 like that these things.

We have to see from the historical record we have to get those parameter. In this example, that this number of occurrences of this petrol pump 4 in 100 kilometer. So, this also we have to see, that entire stretch of entire length of this highway and from there we have to find out what is the data, that is there and then, we can answer this type of probabilistically answer that, what is that at the most 3 petrol pumps in the next100 kilometer and so, that data that we are using for this type of problem can be taken from this, whatever the historical record is available to us, in case of the temporal direction or whatever is available on ground in case of the special direction.

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The central limit theorem can be stated as if X i for there are i equals to 1, to n that there are the n independent and identically distributed random variables. So, this independent and identical distribution, this we may have used earlier in other classes also, this means that this X1, X2, X3,
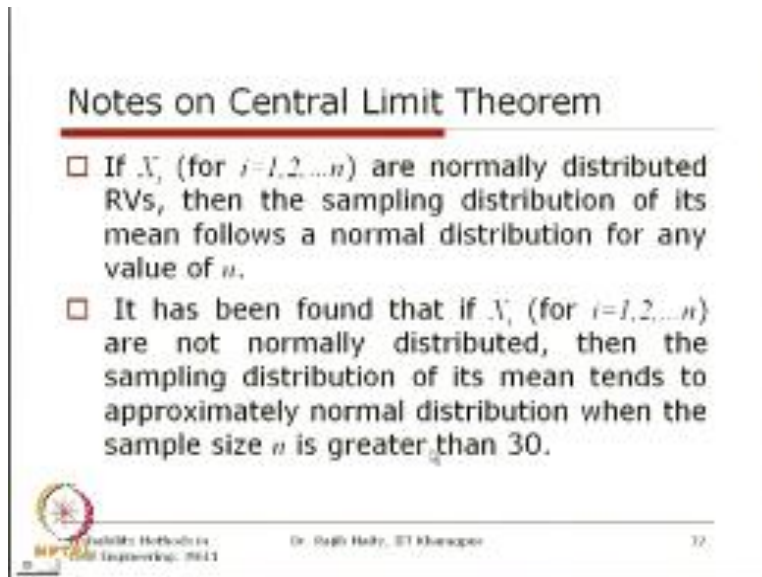
up to X n, these are the random variables, which are independent to each other and their distributions are identical.

And there from a population with the parameters of this μ and sigma square. So, they are all having this parameters the mean is the μ and the variance is this sigma square. Then, the random variable, which is the Xn bar, that is the average of this, all this n random variables, which is we can express at the summation of this all X i divided by n, which is the arithmetic average.

So, this is a new random variable, which we can say that it also tend this tends to a normal distribution with the parameters means, mean equals to μ and variance equals to sigma square by n, as n approaches to∞. Now, I see here we have not defined any particular distribution of this X i' s. So, it can have any distribution, but only thing is that whatever the distribution it is having these are identical distributions that what we have defined, but what is there parent distribution, what is the distribution of their particular distribution, this Xi's, we do not know.

So, even though we do not know that distribution, then their mean can be a normal distribution with this parameters. Now, if we are interested, if we know that the particular random variable, that we are considering, which can be considered to be the mean of the several others random variable, even though we do not know what is the distribution of their of those random variables, we can model this mean as a normal distribution having these parameters μ and sigma square by n.

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And other things if the Xi, so, the X1,X2 up to X n are normally distributed, if we say that this is normally distributed, then the sampling distribution of its mean follow again a normal distribution for any value of n. So, if the parent distributions are normal distribution, then for any values of this n, we can say that their mean will follow the normal distribution, with the parameters that we have discussed now.

Now, it has also been shown that if this Xi' s are not normally distributed with means, this is the most interesting case, even though this Xi' s are not normally distributed, then the sampling distribution of its mean tends to approximately normal distribution, when the sample size n is greater than 30. So, we can safely assume that, this mean of this X1, X2, up to Xn, if that n is greater than 30, then that mean will be approximately normal distribution with the parameters, that we have discussed in the last slide with the mu and sigma square by n.

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Now, let us consider, that a data set of the densities of the concrete from a particular mix, the data set can be assumed to form a normal population and the mean and standard deviations are assumed to be 2450 kg per meters cube and this standard deviation is 25 kg per m^3.

Now, if this is the thing because now you see that, I have given this as the mean and this is as the standard deviation, now because these things, we can cast several such concrete from that particular mix design and we can have their records and those records, if we get their mean and standard deviation we will get this values. So, background to this two data that is 2450 kg per meter cube and 25 kg per meter cube is that that experiment.

So, far as a central limit theorem, as per the central limit theorem, the distribution of this X n bar, that mean of their densities, that is the X i by n will be a normal distribution with parameter mu and sigma square by n is the variance.

So, from the standard normal table that is probabilities that Z is less then equals to 2.575 So, this 2.575, you know that we have referred to this standard normal table several times, this is

respective to this 99 percentage probability, 0.99.5. So, this is a 99 percentage confidence intervals to the right hand side 0.05 and left hand side 0.05 is left.

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So, this one if we see then this X bar minus μ x divided by this standard deviation, that is 25 by this square root of their n, standard deviation by n, this is basically the sigma square when we are talking about this is 25square. So, the standard deviation will be, that 25 square root of 25 square by n. So,25 by square root n, if we take this one, then we can do this one, this one approximately it will become as this Xi - μ x to this 1 and finally, we will get the 0.99 putting this one. So, for the small sample size of25, the 99 percentage confidence interval is estimated mean density lies within this 25 sorry 12.875 kg per meter cube, say 13kg per meter cube of the true value.

This confidence interval will of course, increase with the increase in the standard deviation or decrease in the sample size as you can see it from here. So, this is the true value. So, deviation from the true value for the 25 samples, sample size of 25, if we put that n equals to 25, then it will become as 12.875.

Problem on Central Limit theorem

Q. If a concrete cube has an average strength of 25 KN/m² with a standard deviation of 4 KN/m², what is the probability that a sample mean strength of a sample of 10 such concrete cubes will be between 23 KN/m² and 29 KN/m²?

Quickly, we will see one example, if a concrete cube has an average strength of 25 kilo Newton per meter square, with a standard deviation of 4 kilo Newton per meter square, what is the probability that sample mean strength of a sample of 10 such concrete cubes will be between 23 kilo Newton per meter square and 29 kilo Newton per meter square.

So, this we can get from there their respective means and we can reduce variate, we can get their standard deviation also is known. So, this type of problem, we have shown earlier, how-to refer to the standard normal table and all. So, you can get the probability 0.9421.So, the probability that a sample mean strength of the sample of ten such concrete cubes will be between 23 Kn/m^3 and 29 kn/m^3 is your this 0.9421 which is the difference between these two.

So, in this module, we have seen that there are several examples have been taken and this one. So, the few continuous distribution few discrete distribution we have taken and from next module onwards, we will see some of this statistic, sample statistic based problem how to define that different distribution, how to satisfy which distribution is following and other statistical test we will follow in the next module. Thank you.

Probability Methods in Civil Engineering

End of Lecture 34

Next: "Sampling Distribution and

Parameter Estimation"

in Lecture 35

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