Advanced Hydrology
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Lecture – 5

Good morning and welcome to lecture number 5 of this course on advanced hydrology. In the last class we looked at the over view of the derivation of that Reynolds transport theorem. Then we looked at the reduction of continuity equation from the Reynolds transport theorem and also the momentum equation. Then we moved on to the integral form of the continuity equation, and we wrote down for volumetric continuity equation how the form would be. After that we said that the most of the hydrologic Variables are discrete in nature. So, we need these basic laws of physics on the discrete time domain.

So, we wrote the continuity equation in that discrete time domain it is call the discrete continuity equation. Then we looked at 2 different types of data representation; first one was the sample data representation, and the second one was the pulse data representation. Then we moved on and we looked at or we rather we started looking at one example of momentum equation. This was on a 60 degree elbow. So, what we will do is we will start with that example when we closed in the last class we had said that you should try to attempt or you know try to solve or at least think about how that problem can be done, I am sure some of you may have a try to do that and some of you would have been successful and some others may not have been successful. So, we will start with the solution of that elbow problem today.

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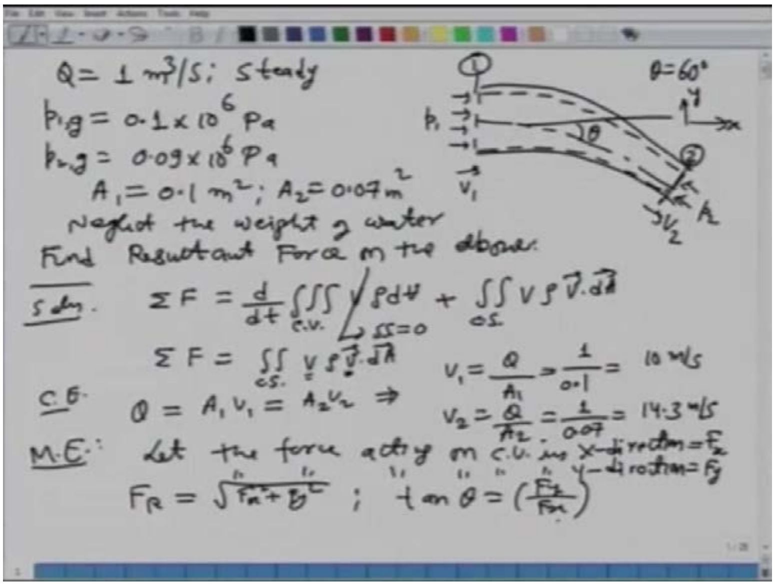

So, we define the problem once again we are looking at 60 degree elbow like this. We draw the control volume which is the dash line and that is right next to the elbow. This elbow is nothing but a pipe fitting. So, there is pressure 1 acting on this side, this pressure 2 acting on this side, the velocity here is V 1 at the inlet. So, this is the direction of flow goes like this, this is the center line of the elbow and beside that it is a 60 degree elbow. So, this angle theta is 60 degrees. And the velocity here comes out perpendicular to the cross section here is V 2.

And then we say that we assumed some coordinate system this says x and y; this is our cross section 1 and if we take the second cross section which is perpendicular to the velocity of flow. The data that are given r the discharge is equal to 1 meter cube per second which is steady state. That is to say that a discharge is flowing at velocity V 1 at cross section a and V 2 at cross section 2. And it is a steady state. The values of the pressure or the gauge pressure at cross section 1 and 2 are given to us or known to us 0.1 mille Pascal is 0.1 Armstrong to the power 6 Pascal’s V 2 g is given as 0.09 mille Pascal’s which is this much also given to us is the area of cross section as 0.1 square meters and the same as cross section 2 is 0.97 square meters.

So, these are all the data that are given to us also given is neglect the weight of water neglecting the weight of the water what you have to find is the resultant force on the elbow. So, this is the problem statement what we do in this case is we make some assumption about the control volume. Then we write down the expressions or we draw what is called a free body diagram in which we mark all the forces or the free dominant forces which are acting on that control volume. Forces are vectors so we will take the components of the forces in the x and y directions in this case. So, we write all of them and then what we do is we use our Reynolds transport theorem for the steady state case which we have just written. So, what we will do is we will start with the Reynolds transport theorem momentum component of that i x and y directions. And then we try to find out how much is the force acting on the control volume in x and y directions. Once we have that we can find out the resultant force magnitude as well as the direction.

So, this is the force which is acting on the control volume. Now, the control volume will exert an equal and opposite force on the elbow. So, the force on the elbow will be just equal and opposite to what we find out. So, let us get started and first write your momentum equation which is summation of the left hand side is equal to your first component is time rate of change of extensive property stored within the control volume which is triple integral beta is V rho d V plus the momentum flowing across all the control surface is beta rho V dot V a. We say that this problem is steady state that is why this first component will be equal to 0. So, what remains with us is this summation of or the net forces that are acting on the control volume should be equal to the net out flux of momentum flowing across all the control surface in this domain which is this.

So, then what we do is we need to find out what will be the velocities at two cross sections. What we do first is we use the continuity equation on this control volume which says that Q is equal to A 1 V 1 is equal to A 2 V 2. I am sure all of you know this. So, that will give us V 1 as Q over A 1 which is 1 over A 1 is 0.1, and that will be 10 meters per second all these data are given to us. And similarly, V 2 would be here Q over A 2 which is 1 over A 2 is the let me see 0.07. So, it will come out to be approximately 14.3 meters per second. So, we have V velocity components, now these velocities are acting perpendicular to the cross section; cross sections 1 and 2. Now, what we do is we apply the momentum equation in x and y directions momentum equation. And before we do that we say that we let the force that is acting on the control volume in x direction is let us say F x and similarly, we say that the force acting on the C V in the y direction is F y.

So, we will account for these forces on the left hand side of this Reynolds transport theorem or A 1 momentum equation. And once we have done that, then your we will come to the computation little later. The resultant but v magnitude of these two components F x square plus F y square and 10 theta or the direction will be equal to F F y over F x . So, if you are able to find this F x and F y we would be able to find the force that is acting on the control volume and equal. And opposite force would be the force acting on the elbow. So, let us try to find these things.

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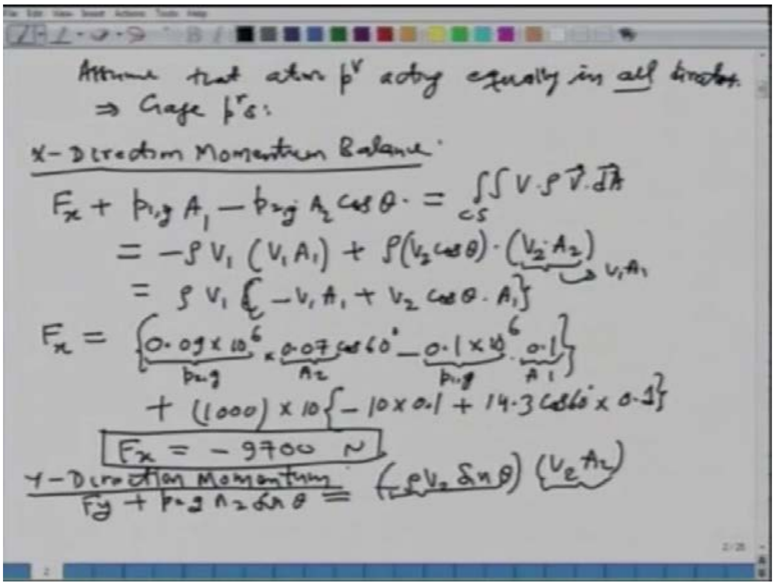

Now, what we will do is we will assume that the pressure force or the atmospheric pressure is acting equally in all directions. So, that will cancel out. So, basically what that mean is we just need to consider the gauge pressure which are given to us then as far as the pressure forces are concern. So, let us write then x direction momentum balance or we write the momentum equation in the x direction. So, let us say V F x was the force acting on the control volume in x direction. What are the other force are acting if you look at your diagram you have pressure force. So, this is the pressure and it is acting over an area of A 1. So, if you look at your diagram this is parallel to the x direction. So, this is a P 1 g A 1. What about the pressure force in the y direction? If you see that it is at the cross section 2 is not acting on the y direction. But it is acting at certain angle. So, what we do is we take the component of that pressure force in the y direction. So, that you can see easily d be p 2 gauge times A 2 and it is component will be cos theta in the y direction.

Now, this ha whole thing should be equal to let me first line I take across the control surface of your V times rho times V dot d A, this whole term at inlet and outlet or at cross section 1 and cross section 2. So, if we did that we write this is going to be rho times V 1 I am taking the rho out and the velocity of flow at cross section 1 is V 1. So, this is V 1 times V dot d A at cross section 1. And what is the V dot d A at cross section 1? It is nothing but V 1 times A 1, we are taking the velocity same or the mean velocity at cross section 1. And we are said that we take the influx always negative. So, we put a negative sign in front, and then you have plus at the outlet; you have the velocity in the y direction.

Now, you see that velocity is going down at an angle at certain angle. So, if we take its component in the y direction V cross theta will get involved. So, you have the velocity itself is V 2 cos theta. And then the flux term will be V dot d A that is velocity V 2 times A 2. And this we can write as rho times V 1 if we take outside, then we will have minus V 1 V 1 plus V 2 cos theta times this V 2 A 2 is basically you can write as V 1 A 1 from the continuity equation and V 1 we have taken out. So, it is A 1. So, either we can do that or we can put all the values of V 1 V 2 A 1 A 2, everything is known in this equation. So, let us then write what will be F of x from this equation? So, it is going to be you are taking the things on the right hand side. So, you have 0.09 10 to the power 6, what is this? This is nothing but e 2 g this is given to us times A 2 which is 0.07 and then you have cos 60 that is V 2 or the component of V 2 and V y direction minus 0 0.1 10 to the power 6. This is your p 1 g and times area is 0.1 which is in square meters which is A 1, this one is not V 2, this is A 2 if you look at this equation p 2 A 2 cos theta and P 1 A 1 we have taken on the right hand side.

So, this is the four terms and similarly, we can write the other things which is going to be plus you have rho of water is that 7000 times V 1, we had found out as 10 meters per second is outside. And then you have minus V 1 A 1 which is 10 times A 1 is 0.1 square meters plus V 2 was 14.3 meters per second. So, we have V 2 and cos of 60 degrees, and then you have A 1 which is 0.1. So, we have all we have done is we have plugged in all the values that are known to us in terms of the pressures, the velocities, and the areas in the momentum equation and after that we can solve it. So, what we will be able to find then is F x is equal to minus 9700 Newtons. Similarly, we write this equation in the y direction or the y direction momentum in which your force acting on the C V is A F y.

What is the component of the pressures force acting in the y direction? It will be p 2 g A 2 sin theta there should be very easy to see for you and equal to the flux term in the y direction. So, it would be until it directly minus V 2 sin theta times V 2 A 2. So, you see that this no term on the right hand side which involves V 1 why is that? It is because the velocity V 1 is parallel to the x direction. So, it is component in the y direction will be 0. So nothing gets contributed to the momentum flux in y direction.

Due to flowing the x direction at cross section 1 it is important to understand that, and then only component we have is from the y direction. And the velocity of flow is in the outward direction, direction positive direction y, direction is in the vertical direction that is y we take this sign as negative. So, this term is negative, because of the direction of flow and you should not get confused that why we out flux of the momentum is taken as negative. It is the velocity which is acting in the negative y direction it is because of that and this component is V dot V a simply V 2 A 2. So, once we have understood that all that is needed then is to like all these values.

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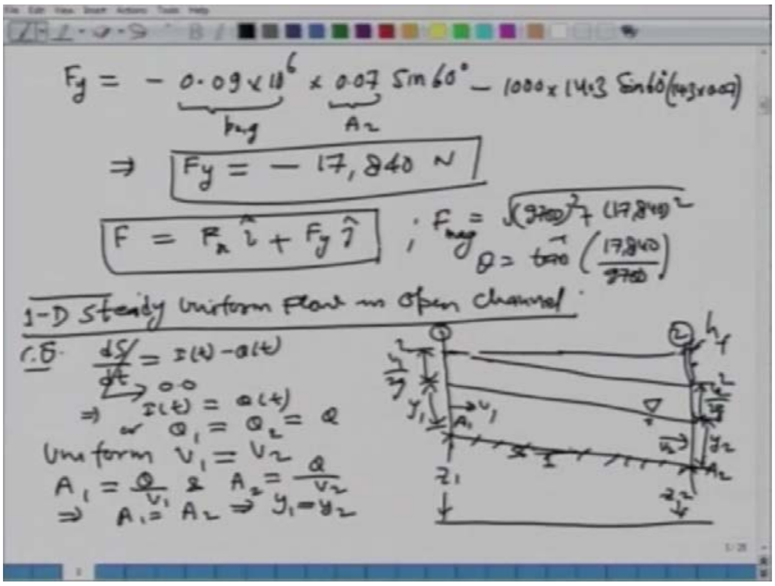

So, we can write down your F of y is equal to it will be minus 0.09 times 10 to the power 6 times 0.07 sin 60 degree. Easy to see that this is your V 2 gauge this is your A 2 and then you have minus rho is 1000. And then you have velocity V 2 which is 14.3 and sin of 60 multiplied by V 2 A 2 which is 14.3 times 0.07. You can simplify this and then what we will get is V component of force acting on the control volume is the y direction as minus of 17840 Newtons. So, we have F x and F y. So, we can say that the force that is acting is F x I in the vector notation plus F of y j. And you know how to find the magnitude and direction of any vector I have written those equations down. So, I am not going to do that simply you can say that the magnitude would be square root of 9700 square plus 17840 square and theta would be 10 inverse of 17840 over 9700. We can find out these numbers I am not going to do that. So, this way we see that the application of the momentum equation to a practical problem of finding a force on the elbow. What we just found is the force acting on the C V and we will take the force acting on the elbow just equal and opposite of that. So, this way we can analyze the simple application or simple problems of momentum finding force finding a areas and so on.

Let us move on and the next this we are going to look at is another example which is also application of your momentum equation. And we are going to use is for a case of open channel. So, it is not only the momentum, but it will involve all the 3 equations. So, we are looking at one dimensional steady uniform flow in open channel. That is what we are going to look at next. So, this is the channel bad, and then you have let us say the water surface elevation in the channel is represented by this. This is your cross section 1; this is your cross section 2, then you have V energy had which is V square by 2 g. And so if we represent everything let us say this says the slope of the channel is s 0, this is a depth of flow at cross section 1 is y 1, the depth of flow at cross section 2 is y 2. And let us say this is the D term which is horizontal or it can be actually any other location.

So, it may be down there as well so this is your z 1 the D terms had and this is z 2. And this is the velocity had which is V 2 square over 2 g, you all are familiar with that and it says V 1 square over 2 g. And what is this? This is call the head looks or Ajax term. So, what we are doing is we are trying to analyze the situation wherein we have a one dimensional steady flow or uniform flow taking place in an open channel. So, what we will do is we will apply all the 3 equations and try to analyze the situation that is continuity equation, momentum equation and also the energy equation in the form of Bernoulli? So, if we apply the continuity equation which we have derived in the form of this and let me also point out that the velocity here is V 1 and the velocity cross section 2 is V 2. The area of cross section is here A 1 and the area of cross section here is A 2. So, if we take the continuity equation and because of the things are being steady. So, this term will be 0. So, what will that involve or tell us that your in flow is equal to out flow or in if let us say if the inflow was Q 1 and outflow was Q 2 all of that we say is equal to Q.

Now, we said that the flow is uniform; uniform flow means that the velocities are same. So, does not change with respect to space. So, we say your V 1 is equal to V 2 what does that mean? That means your area of cross section at one will be over V 1 and area of cross section 2 will be Q over V 2. Now, if let us say assume this is a rectangular channel with the same width it would mean that A 1 is equal to A 2 why because A 1 is equal to Q by V 1 and A 2 is equal to q by V 2 and V 1 is equal to V 2. So, both of these quantities are same, so that means your y 1 is also equal to y 2. What does that tell us that is if the flow is uniform, then the velocities are same with respect to space that is cross section 1 and cross section 2 and also the depth of flow is same that is y 1 is equal to y 2.

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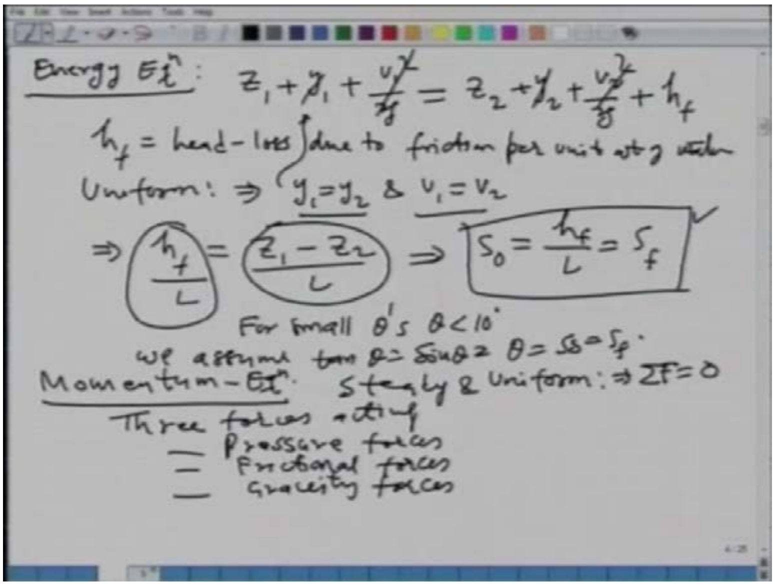

The next thing we are going to look at is the energy equation. It is not the energy equation in terms of the heat energy, but it is more of the potential kinetic and the pressure had right. So, as all of you know the energy equation is represented by this Bernoulli equation in this case z is the potential head, y is the pressure head and V 1 square by 2 g is your kinetic energy. And then you have z 2 plus y 2 plus V 2 square by 2 g at cross section 2 there is some looks and which is due to friction and other we call that h of f that is your h f is the head looks due to friction or unit weight of water in this case the fluid.

So, we apply this energy equation knowing, what are the values of different variables. Now, we have the uniform flow we just saw and application of the continuity equation for the uniform flow we have just found out that y 1 should be equal to y 2 and also V 1 was equal to V 2. So, we put these things these two quantities into this equation so y 1 will cancel out with y 2 and the quantity will cancel out with this quantity so that would simply mean that your h of f is equal to what z 1 minus z 2? Next thing what you do is you divide this equation by let us say the reach length is l on both sides. Then what is h f by r and what is change in the water marivations divided by r? That means your S naught which is the bed slope of the channel is nothing but h f by L which is S of f of which we know already from our knowledge of open channel flow, because this kind of situation.

So, this is how we derive this concept for a uniform flowing channel that the bed slope is equal to the friction slope for uniform flow. Normally for a small thetas that is may be theta 1ess than 10 degrees, we assume 10 theta is equal to 10 theta is equal to you know theta sometimes is equal to o is equal to S f in this case. So, whatever is your angle if it is small, then we can take them all equal to the same thing. So, this is as far as the application of the energy and continuity equation what we are going to next do next is apply the momentum equation to this a flow concept of one dimensional flow to the uniform flowing channel. So, let us look at the momentum equation then for this particular case of one dimensional steady uniform flow in open channel. And we had seen yesterday that for steady and uniform flow what is the momentum equation? Or we reduce form of the Reynolds transport theorem it is nothing but summation of is equal to 0 both the terms and the both the components on the right hand side are equal to 0.

So, what we do then is we look at the predominant forces which are acting on this control volume. So, we take a reach of length L of the channel, and then we draw the control volume which is the water surface elevation the cross sectional areas and the bad. What are the major forces which are acting on this? So, if you think about the situation we should be able to find that there are 3 forces or 3 main forces acting which are those one is the pressure force, other is the weight force or the gravitational force. And the third one is the friction or the resistance force or the viscous force this. So, if we see and then you first write it you have the pressure forces acting on the control volume then you have the frictional forces and the gravity forces. So, what we do is we will write the component of these forces in the x direction, remember it is one dimensional flow. So, our x direction is the direction of flow. So, we will be writing all these forces only in that direction.

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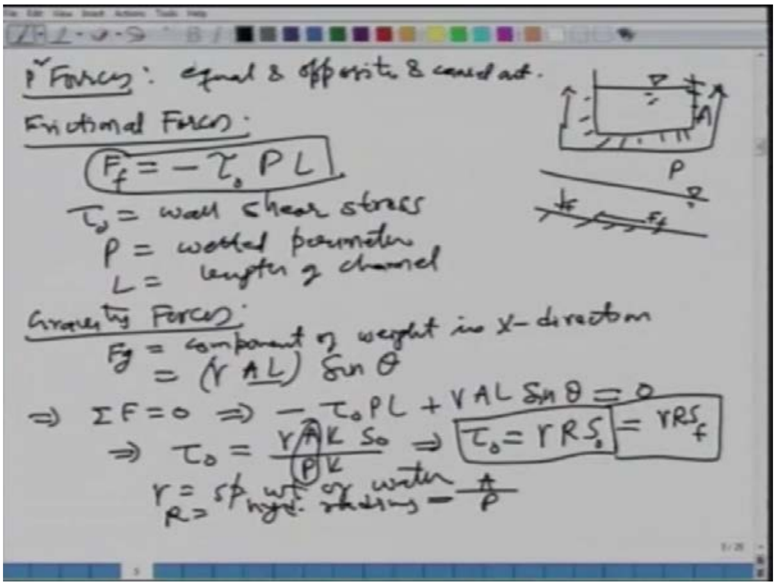

We are moving on. So, if we take let us say this is a channel it has that induct the flow and a cross sectional area is A. And let us say the, this whole quantity what do we call this? This is your vacate perimeter P. So, if we write let us look at each of these courses one by one what are the pressure forces acting on this? We have depth of flow y 1 on the input cross section and we have a depth of flow y 2 and both of them are equal. So, if pressure forces will be equal and opposite that is P 1 A 1 and P 2 A 2 and P 1 will be equal to P 2. So, the pressure forces are equal and opposite on the control surface, equal and opposite and they cancel out. Or we do not even need to worry about the pressure forces in this case. What about the frictional forces?

So, if we have this bed channel and this is your water surface elevation, the frictional forces are the resistance to flow. Due to these bed and sides of the channel, the bed and the side of the channel will resist the flow of water in the channel. Due to that you have the frictional forces which are acting in the opposite direction to the direction of flow. So, that is why this is let us say your F of f what is the magnitude of let us say this frictional forces from our knowledge of the open channel flow hydraulics we will write this as negative of tau 0 e l where your tau 0 is equal to the wall shear stress or it is the shear stress which is causing the resistance to flow that is force per unit of area and the area on which it is acting is nothing but the P times l P is the weighted perimeter and L is length of the flowing question.

So, what is P is the weighted perimeter it will be in the length units and L is of course, the length of the channel which is under consideration. So, this is what it is going to be will leave it at that. And we move on to the gravity forces the gravity forces are acting in vertically downward direction. So, the whole weight of the control volume or the fluid or the water that is contained in the control volume that is acting in the downward direction, but we need to take it is component in the direction of flow which is your x direction. So, let us say that your F of g is the component of weight of the water in x direction what will that be? It should be easy to see that it will be gamma A 1 gamma A 1 is the A 1 is the volume area of cross section times length, gamma is the specific weight of water. So, that is the weight and then it is component in the x direction will be sin theta should be easy to see for you.

So, now what we do is we put these things in the momentum equation. So, that I will give us minus tau 0 P l plus gamma A 1 sin of theta is equal to 0. And all of you should be able to find this tau easily, what is tau 0 will be equal to gamma A 1 for sin theta, we can write as 0 and divided by P l and what is A by P and will cancel out of course, A by P is the hydraulic radius R. So, we have tau 0 is equal to gamma R S 0 and for uniform flow It will be gamma R S f where gamma is the specific weight of the fluid which is water in this case. And R is the hydraulic radius which all of you know is defined as cross sectional area divided by the weighted perimeter. So, what we have done now is that we have derived the momentum equation applicable to the one dimensional flow which is steady and uniform in an open channel.

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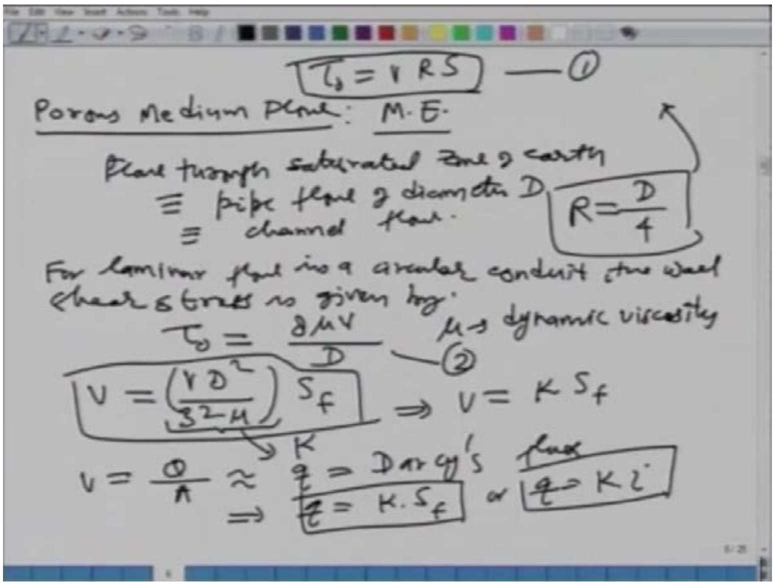

So, what is momentum equation? tau 0 is equal to gamma R S. What we will do next is we will try to extend this consort to ground water flow. And we will derive one of the basic equations which is called the Darcy's law which is very predominantly used in hydrology for analyzing the ground water flow situations in the saturated zone. So, let us look at the porous medium flow situation and what we will do is we will try to apply this momentum equation there. What is happening in the ground water is you have the soil matrix. And there are certain vides alight and what we assume there is that the water is flowing through these vides by what is called this tiny conduits or pipes which are formed due to the vides in the side. So, what we will do is we say that the porous medium flow or flow through saturated portion of earth. We will say let us say equivalent to pipe flow of diameter D let us say that is equivalent to the channel flow which we are just derived for which we have said that tau 0 is equal to gamma R S. For this pipe flow the hydraulic radius it should be easy to see for you all is will be equal to D y 4 I will not derive that you can do that easily.

Now, for laminar flow in a circular conduit, the wall shear stress is given by this equation. We will take this from our knowledge of fluid mechanics I do not expect you to remember this. But we will take this expression from our fluid mechanics tau 0 is equal to 8 mu V over D where what is mu is that dynamic viscosity of the fluid, and V is the velocity of flow and D is the diameter we have already define. Now, what we do is if we compare these two expression, let us say this and this and we also put R into this and combine all this it should be easy to see that the expression for velocity will come out as this gamma D square over 32 mu times S l. Is this equation sound familiar to some of you? Maybe it does not but if I tell you that this whole expression if I say that this is some constant that depends upon the property of the fluid as well as the soil matrix in this saturated zone of R. And let us denote this as K and you know that this K is nothing but the permeability or the hydraulic conductivity of the. So, if we did that then we have V is equal to K times S of in the ground water flow the whole of the cross sectional area is not available for flow.

So, what we do is we say that this is Q over a and we say that this is what is called the Darcy's flux Q. And this is nothing but we say is the Darcy's flux. So, that is to say your Q is equal to K times S f where S f is the friction slope or in other words we say Q is equal to a i and i is the hydraulic gradient this is the form in which most you may have seen this Darcy's law. So, you see that the Darcy's law is nothing but the momentum equation applied to a simplified a situation of ground water flow in the saturated zone. An important aspect which needs to be remembered in the ground water flow situation is that then the water is flowing through the force the hole of the area is not available for flow.

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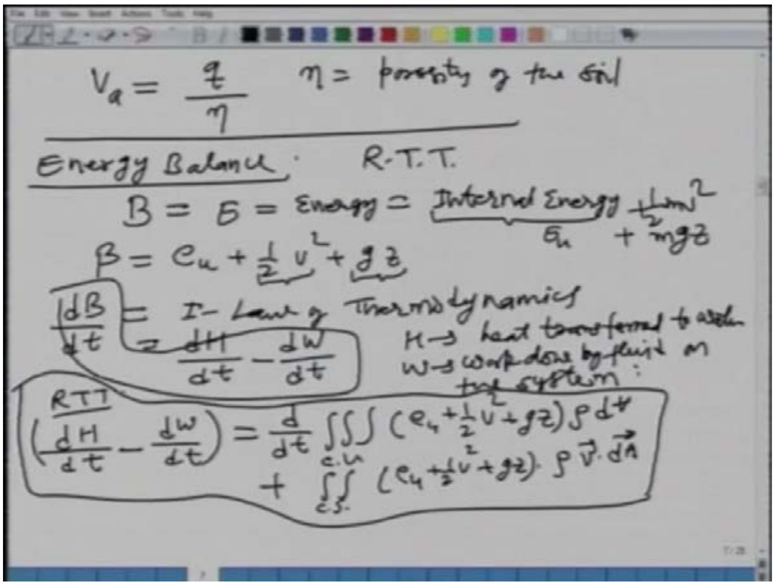

That is why the actual velocity of flow is equal to u divided by eta where eta is the porosity of V 5. So, if we have to find the actual velocity of flow of water through a soil matrix then you find the Darcy's flux and then you need to divide it by the porosity of the soil which will be slightly higher then what you find as the Darcy's flux. So, that will give you the actual velocity which is used in finding out the travel times of tracers. And you know look of this mathematical modeling and management of the act of us.

So, we will not go into the details of those, but this is we application of momentum equation to ground water flow and one dimensional flow to the channel that is what we have seen. Now, what we will do is we will move on and look at quickly what is called the energy, balance or energy equation. When we apply our Reynolds transport theorem and we try to derive the energy equation, then we say that the extensive property is the energy and this will be equal to the sum of 3 different types of energies. We have internal energy plus your kinetic energy which is half M V square plus the potential energy which is let us say m g h or m g z. So, this is your capital V and small v of beta or the intensive energy corresponding to that. And let me say that this is some capital E or e u in fact, so it would be let us say small e u plus half V square plus g times z. So, what we do is we have divided this whole equation by the mass of the fluid. So, this is e u by M will be equal to small e u which is the internal energy per unit mass of your fluid. And similarly, this is your kinetic energy per unit mass of fluid and this is your potential energy per unit mass of the fluid.

Now, what we do is the left hand side of your Reynolds transport theorem is then rate of change of the total energy of the fluid as we have seen in the case of law of conservation of mass and momentum that the left hand side we account for a take care using some external law. In this case what we will do is we will write this as per the first law of thermodynamics. And again we will take this from our knowledge of the thermodynamics, we will not go into the details of that and the first law of thermodynamics states that the rate of change of your total energy of a fluid is nothing but it is equal to d H over d t minus d W over d t where H is the heat transferred into the fluid during the motion or in this case we have water in most of the cases, and W is the work done by the fluid on the system or the surroundings.

So, we will take this equation from the knowledge of our fluid mechanics. Once we have done that then we write the Reynolds transport theorem as d H over d t minus d W over d t that is the left hand side, that is d over d t is equal to again 2 components. First one is the time rate of change of extensive property stored within the control volume this beta rho d V remember which is beta this whole expression e u plus half V square plus g z rho d V. It is the time rate of change of extensive property stored within the control volume which is d by d t of beta rho d V. And the second component is the out flux of the extensive property flowing across the control surface which is beta rho V dot d A and beta is again V u plus half V square plus g z times rho of your V dot d A. So, this is your energy equation for the fluid in terms of the total energy that is possesses. I will leave this internal energy at this point of time, and we will come back to this when we are looking at the evaporation. The next topic is on this transport processes before I move on to that let me look at different types of energies which are existent and we will define this internal energy.

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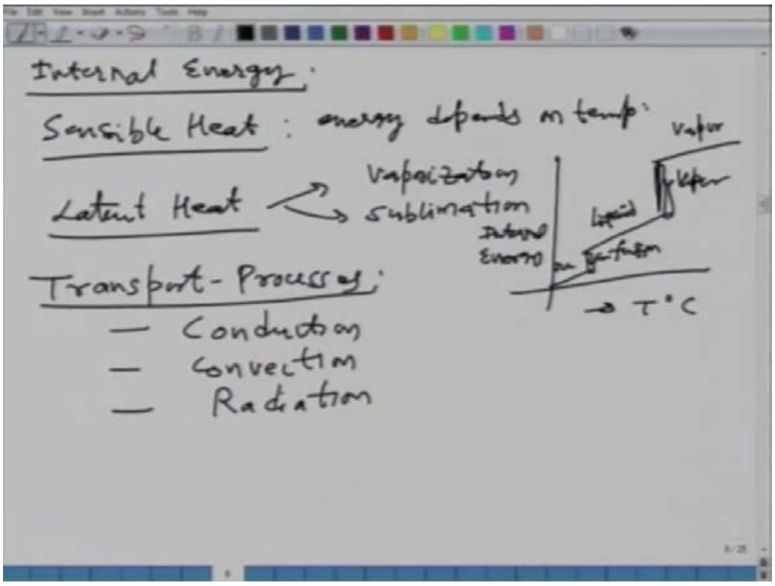

We are all familiar with the kinetic energy and the potential energy that is internal energy can be of 2 types. First one is the sensible heat which is possessed by water ands other is the latent heat. The sensible heat is the energy which depends on the temperature is that component which depends on temperature. And latent heat is something which the fluid possesses without undergoing any changes in the temperature. Again the latent heat can be for vaporization or lot sublimation. And you may have seen this graph in your earlier classes that if try to plot the temperature verses the internal energy per unit volume with respect to minus 20 degree centigrade, what we have is something like this. This is your ice, this is your liquid phase of water. And then you have the vapor here. What is this?

This is the latent heat of vaporization. And this is your latent heat of fusion and what is sublimation? It is the sum of all of these 2.

So, this way we see that there are different types of energies sensible energy and latent heat. And we meet to account for all these different types of energies, then we are talking of this energy equation in this particular how the heat transfer or the transfer of energy takes place? That is your transport processes. I would not like to go into the details of these transport processes, this would be more like a reading assignment for you it is given in the book and you are suppose to go through it on your own except that I will say that there are 3 different ways in which the heat can be transported, what are those?

One is the conduction, other is the convection and third one is the radiation. And I am sure all of you may have seen these 3 different modes of transport of heat. You see that there are 3 different ways of transfer of heat. And there are certain equations which are associated with this we will be needing this little later in this course. So I request you to kindly go through it there is couple of tables which describe the final equations in terms of the momentum transport. You know with similarity of the heat transport. At this point of time I would like stop here. And we complete this chapter and in the next class we will start with the subsurface flow chapter.