Introduction to Aircraft Control System

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Week – 08

Lecture – 40

State-Space Form of Longitudinal Aircraft Equations

In this lecture, we will be dedicating how we can find the linear form of the moment equation along y-axis because the moment along y-axis actually generates the desired pitch angle. Now, let me rewrite the linear form of the forces along x and z direction, then we will start the how we can find the linear model of the moment equation along y-axis.

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w \Delta w + g\Delta\theta \cos\theta_0 = X_{\delta e}\Delta\delta_e + X_{\delta t}\Delta\delta_t \dots\dots Eq(1)$$

$$-Z_u\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[\left(Z_q + u_0\right)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \dots\dots Eq(2)$$

So if you notice these equations are linear form of the forces along x and z direction now we will find the linear form of the moment equation along y direction so where we are going to control the pitch rate of the y-axis and finally considering the third longitudinal equation for $p = r = 0$, so here only we are looking at the pitch rate along y direction so we can write for this condition so here basically this is the linear form of the equation the linear form of the forces along x and z direction now we are going to find the linear form of the moment equation in the longitudinal motion so for this condition $p = r = 0$ we can write the moment equation

$$m_p = I_y\dot{q} \dots\dots Eq(3)$$

Where I_y is moment of inertia along y axis now following the same of including the perturbations variable we get

$$\left(m_p\right)_0 + \Delta m_p = I_y\frac{d}{dt}(q_0 + \Delta q)$$

For steady state trim flight condition, $\left(m_p\right)_0 = q_0 = 0$ .This is basically the trim condition for the longitudinal motion of the aircraft so

$$\Delta m_p = I_y \Delta \dot{q} \ldots\ldots Eq(4)$$

This equation is written in the perturbed variable form and so here Δm_p basically the total moment part of moment acting on the y-axis due to the different variable in the system. So let's see how we can write the Taylor series expansion of Δm_p where the different variables comes into picture, let me write

$$\Delta m_p = f(\Delta u, \Delta w, \Delta \dot{w}, \Delta q, \Delta \delta_e, \Delta \delta_t)$$

So these are the variables which generates the perturbed moment in the y-axis or if you write in taylor series expansion we can write

$$= \frac{\partial m_p}{\partial u}\Delta u + \frac{\partial m_p}{\partial w}\Delta w + \frac{\partial Z m_p}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial m_p}{\partial q}\Delta q + \frac{\partial m_p}{\partial \delta_e}\Delta \delta_e + \frac{\partial m_p}{\partial \delta_t}\Delta \delta_t \ldots\ldots Eq(5)$$

Substituting Eq.(5) in Eq.(4)

$$\Delta \dot{q} = \frac{1}{I_y}\left[\frac{\partial m_p}{\partial u}\Delta u + \frac{\partial m_p}{\partial w}\Delta w + \frac{\partial Z m_p}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial m_p}{\partial q}\Delta q + \frac{\partial m_p}{\partial \delta_e}\Delta \delta_e + \frac{\partial m_p}{\partial \delta_t}\Delta \delta_t\right] \ldots\ldots Eq(6)$$

Denoting $M_u = \frac{1}{I_y}\frac{\partial m_p}{\partial u}$ and son on are the aerodynamic derivatives divided by I_y. Also we know that $\Delta q \approx \Delta \dot{\theta}$. Hence Eq.(6) can be written as

$$-M_u \Delta u - \left[M_{\dot{w}}\frac{d}{dt} + M_w\right]\Delta w + \left[\frac{d^2}{dt^2} - M_q \frac{d}{dt}\right]\Delta \theta = M_{\delta e}\delta_e + M_{\delta t}\delta_t \ldots\ldots Eq(7)$$

So if you notice this equation is written in linear form and here we have another set of linear equations which are Eqs.(1) and (2) so combining these equations yields ODE with constant coefficients. So it means the equation also time invariant because as per the property of time invariant the coefficient of the equation does not vary with time so this system also you can say linear time invariant system or LTI system. These coefficients are made up of aerodynamic stability, mass and inertia characteristic of the aircraft. Now these equations can be written as in state space form $\dot{X} = AX + BU$.

Here X is the state vector and U is the control vector which will be designed through the control algorithm which will provide the desired control input to the system and X can be propagated so here matrices A and B contains the stability derivatives now let's look further how we can find the state space form in this form $\dot{X} = AX + BU$. From Eq.(1)

$$\Delta\dot{u} = X_u\Delta u + X_w\Delta w - g\Delta\theta\cos\theta_0 + X_{\delta e}\Delta\delta_e + X_{\delta t}\Delta\delta_t \dots\dots Eq(8)$$

In Eq. (2) the derivatives Z_q and $Z_{\dot{w}}$ in practice are usually neglected since their contribution is very minimal in the aircraft response, hence

$$\Delta\dot{w} = Z_u\Delta u + Z_w\Delta w + u_0\Delta q - -g\sin\theta_0\,\Delta\theta + Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \dots\dots Eq(9)$$

Using

$$\Delta q \approx \Delta\dot{\theta} \dots\dots Eq(11)$$

and $\Delta\dot{q} \approx \Delta\ddot{\theta}$,Eq. (7) can be written as

$$\Delta\dot{q} = M_u\Delta u + M_{\dot{w}}\Delta\dot{w} + M_w\Delta w + M_q\Delta\dot{\theta} + M_{\delta e}\delta_e + M_{\delta t}\delta_t \dots\dots Eq(12)$$

Substituting Eq.(9) in Eq.(12) we get

$$\begin{aligned}\Delta\dot{q} = {} & [M_u + M_{\dot{w}}Z_u]\Delta u + [M_w + M_{\dot{w}}Z_w]\Delta w + \left[M_q + M_{\dot{w}}u_0\right]\Delta q \\ & - M_{\dot{w}}g\sin\theta_0\Delta\theta + \left[M_{\delta e} + M_{\dot{w}}Z_{\delta_e}\right]\Delta\delta_e \\ & + \left[M_{\delta t} + M_{\dot{w}}Z_{\delta_t}\right]\Delta\delta_t \dots\dots Eq(13)\end{aligned}$$

This is the final linear moment equation. So we can now write the state space form $\dot{X} = AX + BU$ form. Eqs.(8) (9) (12) and (13) if we write in compact form, then

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\cos\theta_0 \\ Z_u & Z_w & u_0 & -g\sin\theta_0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & -M_{\dot{w}}g\sin\theta_0 \\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_{\delta e} & X_{\delta t} \\ Z_{\delta_e} & Z_{\delta_t} \\ M_{\delta e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta t} + M_{\dot{w}}Z_{\delta_t} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_e\\ \Delta\delta_t\end{bmatrix}$$

So basically if you draw the control block diagram

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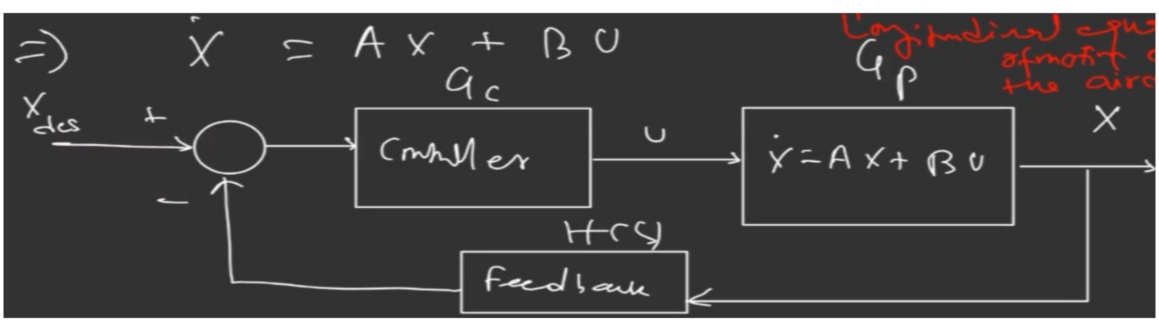

We have this is our reference point where X_{des} are basically the desired value of u, w, q, θ so this is the linear system, so here we are having the controller algorithm basically controller which gives the desired control input to the plant and this is our actual output X and this is feedback path and we have this is the output track by feedback element and this is negative this is positive so that we can if you say this is the powerplant G_p this is the controller G_c and this is H(s) this is basically how the control system closed loop control system can be designed to control the desired value and this is the linearized model of the longitudinal equation of motion of the aircraft. So let's stop it here, from the next lecture we will look at how to find the linearized model of the lateral directional motion of the aircraft. Thank you.